Two problems concerning uniform polynomial approximation of continuous functions

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Abstract

We remind two theorems closely connected with the fundamental P.L.Chebyshev's theorem on the best approximation of functions by polynomials, namely S.N.Bernstein's theorem ([3], p. 292) on reconstruction of a function by its deviations from polynomials, and the author's one on distribution of Chebyshev's alternance points. In connection with this two results two open (in author's opinion) problems are formulated.

Let $C[0,\pi]$ be the Banach space of all real continuous functions $f(t)$ with the usual uniform norm

$$\|f\| = max\{|f(t)| : 0 \leq t \leq \pi\} \tag{1}$$

The following fundamental statement belongs to P.L. Chebyshev ([1], pp. 66 - 70)

Chebyshev's Theorem For every $f \in C[0,\pi]$ and every $n \geq 0$ among all even trigonometrical polynomials P_n of degree n there is a unique polynomial f_n with the minimal deviation from f:

$$E_n(f) = \|f - f_n\| \leq \|f - P_n\|. \tag{2}$$

This polynomial of the best approximation is completely characterized by the following property: the number of consecutive points $t_k^n \in [0,\pi]$ in which the difference $f(t) - f_n(t)$ takes the value $E_n(f)$ with alternating signs (*points of Chebyshev's alternance*) is greater or equal $n+2$.

It's clear that the deviations $E_n(f)$ form a non-decreasing sequence, which due to Weierstrass theorem ([1], p. 41) tends to 0.

Let us note that in the statement of Chebyshev's Theorem one can find not only the original function f and approximating polynomials $\{f_n\}_0^\infty$, but also the sequence E of deviations

$$E_0(F) \geq E_1(F) \geq E_2(F) \geq \dots, \lim_{n\to\infty} E_n(F) = 0$$

and the matrix T of Chebyshev's alternance points

$$T(f) = \{t_n^k, 0 \leq k \leq n+1, n = 0, 1, 2, \dots\}$$

In 1930 on the First Congress of Soviet Mathematicians in Kharkov S.N.Bernstein posed in particular a problem on reconstruction of a function by its sequence E of deviations ([2], pp. 500 - 519). Later ([3], pp. 292 - 294) he have made a more detaled statement of the problem, have solved it but leaved open the question of uniqueness of the function (the posing of the uniqueness problem appeared to be possible only after the detaled statement mentioned above).

Because the problem has geometrical (in Banach space sence) nature, let me formulate it in corresponding language.

Bernstein's problem. Let X be a real infinite-dimensional separable Banach space with a fixed lineary independent complete system of elements

$$x_0, x_1, x_2, \dots, x_n, \dots, \tag{3}$$

and the corresponding increasing system of subspaces

$$\{0\} = X_0 \subset X_1 \subset X_2 \subset \dots; X_n = [x_k]_{k=o}^{n-1}. \tag{4}$$

For every $y \in X$ one can define a sequence of deviations

$$E_n(y) = \min_\lambda \|y - \sum_{k=0}^{n-1} \lambda_k x_k\|, n = 0, 1, 2, \dots \tag{5}$$

Evidently,

$$\|y\| = E_0(y) \geq E_1(y) \geq E_2(y) \geq \ldots, \lim_{n\to\infty} E_n(y) = 0$$

A "polynomial" $y_n = \sum_{k=0}^{n-1} \lambda_k^n x_k$ on which the minimum in (5) is attained is said to be a polynomial of best approximation. Let us demand that for every $y \in X$ and every n there is a unique polynomial of best approximation. In such a case the system of elements (3) (and the corresponding system of subspaces (4)) is said to be a Chebyshev system (T-system).

Let now X be a Banach space with a T-system $\{x_n\}_0^\infty$. For every $y \in X$ with non-zero deviation $E_n(y)$ let us denote by m_n the smallest $m \geq n$ for which the coefficient λ_m^{m+1} of the polinomial y_n of best approximation is non-zero. Then for every non-zero deviation $E_n(y)$ let us assign the sign

$$signE_n(y) = sign\lambda_{m_n}^{m_n+1} \tag{6}$$

Bernstein's Theorem For every sequence $\{\alpha_n\}_0^\infty$ of reals which sattisfies conditions 1. $|\alpha_n| \geq |\alpha_{n+1}|$, 2. if $|\alpha_n| = |\alpha_{n+1}|$ then $\alpha_n = \alpha_{n+1}$, and 3. $\lim_{n\to\infty} \alpha_n = 0$, there exists an element y, for which

$$E_n(y) \times signE_n(y) = \alpha_n, n = 0, 1, 2, \ldots \tag{7}$$

In particular the theorem holds true for $X = C[0, \pi]$ with T-system $\{\cos nt\}$.

It is natural to call a T-system for which one has the uniqueness in Bernstein's theorem a Bernstein system (B-system).

In general setting the following two facts are known:

Theorem (M.I.Kadets [4]). The set of all elements $y \in X$, which are defined uniquely by their deviations from a fixed T-system, is a dense G_δ-set.

Example (R.G.Long [6]). There exists an eqiuvalent norm on the space c_0 in which the canonical basis forms a a T-system which is not a Bernstein system.

Let us formulate at last the first question.

Problem 1. Is it true that every function $f \in C[0, \pi]$ is determined uniquely by its deviations from T-system $\{\cos nt\}$? In other words, is it true that the T-system $\{\cos nt\}$ in $C[0, \pi]$ is a Bernstein system?

The next question deals with the the matrix T of Chebyshev's alternance points. In 1960 I've made an attempt to clarify what one can say about this matrix for an arbitrary continuous function f. The following result was obtained:

Theorem [5]. For every $f \in C[0,\pi]$ and every $\varepsilon > 0$

$$\liminf_{n\to\infty} \Delta_n n^{\frac{1}{2}-\varepsilon} = 0, \tag{8}$$

where $\Delta_n = max_k \left| t_k^n - \frac{\pi k}{n+1} \right|$.

Problem 2. What improvements of (8) one can obtain without additional restrictions on f's smoothness?

References

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