SOME FUNCTIONALS OVER A COMPACT MINKOVSKII SPACE

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In this paper we obtain estimates which are order-exact for the projection and Macphail constants of an arbitrary n-dimensional Banach space: $1 \leq \lambda(X) \leq \sqrt{n}$, $1/n \leq \mu_1(X) \leq 1/\sqrt{n}$.

The collection of all n-dimensional Banach spaces can be metrized with the aid of the Banach-Mazur distance (see [1], p. 216):

$$\rho(X; Y) = \ln d(X; Y), \quad d(X; Y) = \inf_T \|T\| \|T^{-1}\|, \tag{1}$$

where T passes over all isomorphisms of X onto Y. In this way we obtain a compact metric space. We will refer to it as a compact Minkovskii space and denote it by $\mathfrak{M}_n$.

Let Z be any Banach space, and X a subspace of it. We will define the relative projection constant

$$\lambda(X; Z) = \inf \|P\|,$$

where P passes through all projections of Z onto X. We now define the (absolute) projection constant [2]:

$$\lambda(X) = \sup_Z \lambda(X; Z), \tag{2}$$

where Z passes through every Banach space containing X as a subspace. The upper bound in (2) is attained if in the capacity of Z we substitute C(Q), the space of all continuous functions defined on some compact set Q

$$\lambda(X) = \lambda(X; C(Q)) \qquad (X \subset C(Q)). \tag{3}$$

The Macphail constant is defined as:

$$\mu_p(X) = \inf_{\{x_i\}} \sup_{\alpha_i = \pm 1} \left\| \sum \alpha_i x_i \right\|, \tag{4}$$

where the lower bound is taken over all finite subsets $\{x_i\} \subset X$, satisfying the condition

$$\sum \|x_i\|^p = 1 \qquad (p \geqslant 1).$$

The projection constant has bound application in the estimation of compact spaces of large dimension in Banach spaces, and in the theory of completely continuous operators (see [3], p. 206-208 where essential use is made of the estimate $\lambda(X) \leq n$). The constant $\mu_p(X)$ has arisen in connection with studies of unconditionally convergent series.

The constant has been calculated for certain spaces. For example,

$$\lambda(l_1^{(n)}) = \frac{(2k-1)\Gamma\left(k - \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k)}, \quad k = \left[\frac{n+1}{2}\right] \text{ [4], [2],}$$

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$$\lambda(l_2^{(n)}) = \frac{n\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)} \sim \sqrt{\frac{2n}{\pi}} \quad [4], [2], [5],$$

$$\lambda(l_\infty^{(n)}) = 1, \quad \lambda(X) > 1 \; (X \neq l_\infty^{(n)}) \quad \text{(see [6], p. 160)},$$

$$n\mu_1(l_\infty^{(n)}) = \lambda(l_\infty^{(n)}), \quad n\mu_1(l_2^{(n)}) = \lambda(l_2^{(n)}) \quad [5].$$

In this note we will prove the additional inequalities

$$1 \leqslant \lambda(X) \leqslant \sqrt{n}, \quad \frac{1}{n} \leqslant \mu_1(X) \leqslant \frac{1}{\sqrt{n}} \qquad (X \in \mathfrak{M}_n). \tag{5}$$

Previously, the more inexact estimates

$$1 \leqslant \lambda(X) \leqslant \frac{\lambda(l_2^{(n)})}{\sqrt{n}} n \;\; [2], \quad \frac{\lambda(l_2^{(n)})}{n\sqrt{n}} \leqslant \mu_1(X) \leqslant \frac{4}{\sqrt[4]{n}} \;\; [5]$$

were known.

Upper Estimate for $\lambda(X)$. Let U^* be the unit sphere of the adjoint space X^*. In the space $C(U^*)$ we will examine the subspace $L(U^*)$ of all linear homogeneous functions. Each such function is given by the formula

$$F(f) = (f;\ x_0), \ \|F\| = \|x_0\| \qquad (f \in U^*,\ x_0 \in X),$$

which allows the identification of $L(U^*)$ with X. In agreement with (3) the norm of any projection operator, transforming $C(U^*)$ into X, is an upper bound on $\lambda(x)$. Thus, the problem reduces to that of selecting the projection operator well.

We will make use of a proposition due to F. John [7]. In a somewhat modified form it states the following:

LEMMA. For any $X \in \mathfrak{M}_n$ there exists a linear operator $T\colon X \to l_2^{(n)}$ having the following properties: $\|T^{-1}\| = 1$; there exists a sequence of elements $\{y_r\}_1^s \subset l_2^{(n)}$ and a sequence $\{\lambda_r\}_1^s$ ($n \le s \le n(n+1/2)$ such that $\|y_r\| = \|T^{-1}y_r\| = \|T^*y_r\| = 1$. For any $u, v \in l_2^{(n)}$ we find the identity

$$\sum_{r=1}^{s} \lambda_r (y_r,\ u)(y_r,\ v) = (u;\ v), \tag{6}$$

from which we conclude in particular, that $\sum \lambda_r = n$.

We will define the required projector by means of the formula

$$PF = \sum_{r=1}^{s} \lambda_r F_r(T^* y_r)\, T^{-1} y_r \qquad (F \in C(U^*)). \tag{7}$$

It is easily shown that this transformation does indeed map $C(U^*)$ onto $L(U^*) = X$. We will show that the operator P leaves each element of X invariant. Let us choose a function $F_0(f) = (f;\ x_0) \in L(U^*)$ and apply the operator P to it:

$$PF_0 = \sum_{r=1}^{s} \lambda_r (T^* y_r;\ x_0)\, T^{-1} y_r = \sum_{r=1}^{s} \lambda_r (y_r;\ Tx_0)\, T^{-1} y_r.$$

Applying to PF_0 any linear functional $\varphi \in X^*$, and using the relation (6):

$$(PF_0;\ \varphi) = \sum_{r=1}^{s} \lambda_r (y_r;\ Tx_0)(T^{-1}y_r;\ \varphi) = \sum_{r=1}^{s} \lambda_r (y_r;\ Tx_0)(y_r;\ T^{*-1}\varphi) = (Tx_0;\ T^{*-1}\varphi) = (x_0,\ \varphi) = F_0(\varphi),$$

i.e., $PF = F$ for every $F \in L(U^*)$. Thus, the operator P is indeed a projection operator from $C(U^*)$ onto X. We can estimate the norm of the projector P:

$$\|P\| = \sup_{|F(f)| \leqslant 1} \sup_{\|f\| \leqslant 1} \left| \sum_r \lambda_r F(T^* y_r)(T^{-1} y_r;\ f) \right|.$$

Since $\|T^*y_r\| = 1$, we can set $F(T^*y_r) = \text{sign}(T^{-1}y_r; f)$;

$$\|P\| = \sup_f \sum \lambda_r |(T^{-1}y_r; f)| = \sup_f \sum \lambda_r |(y_r; T^{*-1}f)| \leq \sup_f \sqrt{\sum \lambda_r} \sqrt{\sum \lambda_r (y_r; T^{*-1}f)^2}.$$

In agreement with (6)

$$= \|P\| \leq \sqrt{n} \sup_f \|T^{*-1}f\| = \sqrt{n} \|T^{-1}\| = \sqrt{n}.$$

Thus, $\lambda(X) \leq \sqrt{n}$ for every $X \in \mathfrak{M}_n$. As shown by the example of $l_2^{(n)}$, this estimate is close to ideal.

Upper Estimate for $\mu_1(X)$. As proved in [8], the constants $\lambda(X)$ and $\mu_1(X)$ are related by means of the relation

$$n\mu_1(X) \leq \lambda(X),$$

from which we immediately see that $\mu_1(X) \leq n^{-1/2}$.

Lower Estimate for $\mu_1(X)$. For any given $\varepsilon > 0$ we will find a set $\{x_i\}_1^m \subset X$, for which

$$\mu_1(X) \geq \frac{\max\limits_{\alpha=\pm 1} \|\sum \alpha_i x_i\|}{\sum \|x_i\|} - \varepsilon. \tag{8}$$

We will represent the vector x_i with respect to the basis of Auerbach (see [1], p. 213):

$$x_i = \sum_{j=1}^n a_{ij} e_j, \quad \max_j |a_{ij}| \leq \|x_i\| \leq \sum_j |a_{ij}| \qquad (1 \leq i \leq m).$$

A lower estimate is calculated in the right hand side of (8), for the numerator:

$$\max_{\alpha_i=\pm 1} \left\| \sum_i \alpha_i \sum_j a_{ij} e_j \right\| \max_{\alpha_i=\pm 1} \max_j \left| \sum_i \alpha_i a_{ij} \right| = \max_j \sum_i |a_{ij}|.$$

An upper estimate for the denominator is given by:

$$\sum \|x_i\| = \sum_i \left\| \sum_j a_{ij} e_j \right\| \leq \sum_i \sum_j |a_{ij}|.$$

Thus

$$\mu_1(X) \geq \frac{\max\limits_j \sum_i |a_{ij}|}{\sum_j \sum_i |a_{ij}|} - \varepsilon \geq \frac{1}{n} - \varepsilon.$$

By virtue of the arbitrariness of ε we obtain $\mu_1(X) \geq n^{-1}$. This estimate is exact since $\mu_1(l_\infty^{(n)}) = n^{-1}$.

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