ON CONNECTIONS BETWEEN VARIOUS FORMS OF ALMOST-PERIODIC REPRESENTATIONS OF GROUPS

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By a representation of a topological group G into a Banach space B we mean a homomorphism T: G → Aut B, strongly continuous in the sense that all the orbits g → T(g)x (g ∈ G, x ∈ B) are continuous; Aut B is the group of all bounded and boundedly invertible linear operators B → B. A representation T is said to be almost periodic (a.p.) if all the orbits are relatively compact, weakly a.p. if all the orbits are relatively weakly compact, and scalarly a.p. if all the scalar functions† $\tau_{f,x}(g) = f(T(g)x)$ (x ∈ B, f ∈ B*) on the group G are almost periodic.

We recall that a continuous bounded function $\varphi(g)$is said to be right a.p. if the family of its right shifts $\varphi_h(g) = \varphi(gh)$ (h ∈ G) is relatively compact in the uniform metric. In a similar manner one defines the concept of left a.p. but, in fact, by Maak's well-known theorem, these two concepts of a.p. are equivalent and, therefore, one can simply say a.p. With each group G there is associated a compact group $\bar{G}$ (the Bohr compactum of the group G) and a continuous homomorphism R: G → $\bar{G}$ with a dense range such that each a.p.f. $\varphi(g)$ can be extended to a continuous function $\bar{\varphi}$ on G: $\varphi(g) = \bar{\varphi}(R(g))$. This Bohr extension is an isometric isomorphism of the Banach algebra AP(G) of a.p.f. and the Banach algebra C($\bar{G}$) of continuous functions. For any a.p. representation T of a group G there exists a unique representation $\bar{T}$ of the Bohr compactum $\bar{G}$ (into the same Banach space B) such that T = $\bar{T}$R. Conversely, if $\bar{T}$ is some representation of the Bohr compactum $\bar{G}$, then T = $\bar{T}$R is an a.p. representation of the group G. These and further necessary facts on Banach representations can be found in [1]. Also there it is proved that if the space B is reflexive, then any scalarly a.p. representation is a.p. (obviously, the converse is valid in any Banach space). The requirement of reflexivity is essential, as shown by the following.

Example. In the space c of convergent complex sequences $x = (\xi_k)_1^\infty$ we consider the operator A of multiplication by a convergent sequence $\lambda = (\lambda_k)_1^\infty$. We consider the representation $t \to e^{iAt}$ of the additive group **R**. It is scalarly a.p. but, as we show now, it is not even weakly a.p. if λ_k (k = 1, 2, 3,...) are linearly independent over the field of rational numbers. Indeed, in this case the weak closure of the orbit of the point u = (1, 1,...) is the set of all $x = (\xi_k)_1^\infty \in c$ such that $|\xi_k| = 1$ (k = 1, 2, 3,...). It is not weakly compact since, setting

$$\xi_k^{(n)} = \begin{cases} (-1)^k & (1 \leqslant k \leqslant n) \\ 1 & (k > n), \end{cases}$$

we obtain the sequence $u^{(n)} = (\xi_k^{(n)})_{k=1}^\infty \in c$, from which one cannot extract a sequence, weakly convergent in c (the sequence $\{u^{(n)}\}_{n=1}^\infty$ w*-converges to $((-1)^k)_{k=1}^\infty$ in the space m of all bounded sequences).

Remark. If the representation T is weakly a.p. or scalarly a.p., then it is bounded:

$$c_T \equiv \sup_g \| T(g) \| < \infty.$$

In a reflexive space every bounded representation is weakly a.p., but not necessarily scalarly a.p. For example, if A is a bounded self-adjoint operator in a Hilbert space and the system of eigenvectors of the operator A is incomplete, then the unitary representation $t \to e^{iAt}$ of the group **R** is not scalarly a.p.

THEOREM. In every Banach space B, a representation T, which is simultaneously scalarly and weakly a.p., is a.p.

The proof is carried out, with appropriate modifications, by the same scheme as in the reflexive case.

†They are called the generalized matrix elements of the representation T.

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We consider the Bohr extension $\overline{\tau_{f,x}}$ of the generalized matrix elements. Since $\|\overline{\tau_{f,x}}\| \leqslant c_T \|f\| \|x\|$, for fixed $x \in B$, $\gamma \in \bar{G}$ the quantity $\overline{\tau_{f,x}}(\gamma)$ is a continuous linear functional on f. Consequently,

$$\overline{\tau_{b,x}}(\gamma) = (\tilde{T}(\gamma)x)(f), \tag{1}$$

where $\tilde{T}(\gamma)$ is a continuous homomorphism $B \to B^{**}$; the function $\gamma \to \tilde{T}(\gamma)x$ is continuous if B^{**} is provided with the w*-topology. From the identity

$$(\tilde{T}(R(g))x)(f) = \overline{\tau_{f,x}}(R(g)) = \tau_{f,x}(g) = f(T(g)x)$$

there follows that $\tilde{T}(g)x = T(R(g))x$ for all $g \in G$, $x \in B$. Since Im R is dense in the Bohr compactum $\bar{G}$, it follows that $\tilde{T}(\gamma)x$ is for any $\gamma \in \bar{G}$ a w*-limit point of the orbit T(g)x. But, by assumption, this orbit is relatively weakly compact, i.e. its weak closure in B is w*-compact as a subset in B^{**} and, therefore, it is w*-closed. Consequently, the w*-closure of the considered orbit in B^{**} lies in B. But then $\tilde{T}(\gamma)x \in B$ for all $\gamma \in \bar{G}$, i.e. $\tilde{T}(\gamma)$ is an operator in B (obviously, bounded). Since $T = \tilde{T}R$, it remains to verify that $\tilde{T}$ is a representation of the Bohr compactum. This is obtained by a word-for-word repetition of the corresponding arguments from [1] on the basis of the formula (1), which now can be written in the form $\overline{\tau_{f,x}}(\gamma) = f(\tilde{T}(\gamma)x)$.

LITERATURE CITED

1. Yu. I. Lyubich, Introduction to the Theory of Banach Representations of Groups, Birkhäuser, Basel (1988).

ON THE STRUCTURE OF A SET OF SPECIAL SUBCLASSES IN THE SOLUTION OF THE ONE-DIMENSIONAL AND TWO-DIMENSIONAL MOMENT PROBLEMS

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Let $\{a_{nm}\}_{n,m=0}^{\infty}$ be a two-dimensional moment sequence, i.e.,

$$a_{nm} = \iint_{-\infty}^{\infty} \lambda^n \mu^m \sigma(d\lambda;\ d\mu),\quad \sigma(d\lambda;\ d\mu) \geqslant 0. \tag{1}$$

The collection of all measures, defining the representation (1), will be denoted by Σ_a. The object of our consideration consists of those sequences a_{nm} for which there exist measures in Σ_a, possessing an a priori localization, i.e. for which the prescribed domains are free of mass. If D is some domain in $\mathbf{R}^2$, then by Σ_{exD} (Σ_{inD}) we shall denote the subcollections of Σ_a, characterized by the conditions supp $\sigma \subseteq \mathbf{R}^2 \setminus D$ (supp $\sigma \subseteq D$). Having in mind the a priori localization of a measure, it is natural to consider the questions of existence, uniqueness, and description of the set of solutions, which are traditional in the moment problem. We shall use the approach developed in [1] (the one-dimensional part) and [2, 3] (the reduction of the two-dimensional problem to a collection of one-dimensional ones). With the purpose of reducing the number of conditions to be verified, we have modified somewhat the constructions of [1]. We shall adhere to the terminology adopted in [4]. We mention that in the interesting survey [5] there exist results on the localization of the representing measure, which are of a different nature than those given below.

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