RESOLVING AND STRICTLY RESOLVING REGULARIZERS

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Let X and Y be Banach spaces and assume that a continuous (not necessarily linear) injective operator A acts from X into Y. A sequence of continuous operators R_n: $Y \to X$ is said to be regularizing (with respect to A^{-1}) if for each $x \in X$ we have the limit equality

$$\lim_{n\to\infty} R_n Ax = x. \tag{1}$$

From the definition of a regularizing sequence (in the sequel we shall call it a regularizer) there follows directly the following inclusion: the domain $M(R_n) = \{y \in Y: \exists \lim R_n y\}$ of the convergence of the regularizer contains AX. According to Domanskii's definition [1], a regularizer is said to be resolving if $M(R_n) = AX$. The meaning of the terminology is: the convergence of the sequence $R_n y$ to the element x is equivalent to the solvability of the equation $Ax = y$.

<u>THEOREM 1.</u> Assume that the operator A: $X \to Y$ admits a regularizer $(\bar{R}_n)_1^\infty$. If there exists a continuous injection V: $Y \to X$, $V(0) = 0$, then there exists also a resolving regularizer, which will be denoted by $(\bar{R}_n)_1^\infty$. Moreover, if V, A, and R_n are linear, then also $\bar{R}_n$ are linear. If, furthermore, the R_n are finite-dimensional, then also $\bar{R}_n$ are finite-dimensional.

<u>Proof.</u> In the first two cases the resolving regularizer is uniquely constructed:

$$\bar{R}_{2n-1} = R_n, \quad \bar{R}_{2n} = R_n + V(AR_n - I), \quad n = 1, 2, \ldots. \tag{2}$$

Here I is the identity operator. First we show that $(\bar{R}_n)_1^\infty$ is a regularizer. Since $\bar{R}_{2n-1} = R_n$, we have $\lim \bar{R}_{2n-1}Ax = x$. Further, $\bar{R}_{2n}Ax = R_nAx + V(Ar_nAx - Ax)$. In view of the continuity of the operators V and A, we have

$$\lim_{n\to\infty} V(AR_nAx - Ax) = V(A \lim R_nAx - Ax) = 0.$$

Thus, $\lim \bar{R}_nAx = x$. We show that $(\bar{R}_n)_1^\infty$ is a resolving regularizer. Let $\bar{R}_n y \to x$. By virtue of the first of the equalities (2), we have $R_n y \to x$. The second equality (2) leads to $\lim V\cdot(Ar_n y - y) = 0$; but, on the other hand,

$$\lim_{n\to\infty} V(AR_n y - y) = V(A \lim R_n y - y) = V(Ax - y).$$

By virtue of the injectivity of the operator V we have $y = Ax$, proving the fact that the regularizer $(\bar{R}_n)_1^\infty$ is resolving. Now we consider the third case: V, A, R_n are linear operators, while the R_n are also finite-dimensional. In this case a finite-dimensional resolving regularizer is constructed in the following manner:

$$\bar{R}_{2n-1} = R_n, \quad \bar{R}_{2n} = R_n + R_nAV(AR_n - I), \quad n = 1, 2, \ldots. \tag{3}$$

We show that $(\bar{R}_n)_1^\infty$ is a regularizer. As in the previous two cases, $\bar{R}_{2n-1}Ax = R_nAx \to x$. From the second of the equalities (3) we obtain

$$\|\bar{R}_{2n}Ax - R_nAx\| \leqslant \|R_nA\|\|V\|\|A\|\|R_nAx - x\|.$$

Since by the Banach–Steinhaus theorem the numbers $\|R_nA\|$ are bounded in their totality, it follows that the right-hand side of the last inequality tends to zero when $n \to \infty$. Thus, $\lim \bar{R}_nAx = x$. We consider the solvability. Let $\bar{R}_n y \to x$. Then also $R_n y \to x$, while

$$\lim \bar{R}_{2n} y = \lim R_n y + \lim (R_nA) VA(R_n y - x) + \\ + \lim (R_nA) V(Ax - y) = x + 0 + V(Ax - y).$$

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Thus, by virtue of the injectivity of the operator V, we have y = Ax.

We elucidate how important is the requirement of the existence of the injection $V \to X$. We restrict ourselves to the consideration of a separable infinite-dimensional space X. In the nonlinear case the existence of an injection is guaranteed by the following condition: the dimension (the minimal cardinality of a dense subset) of the space Y does not exceed the cardinality of the continuum. This condition can be easily obtained on the basis of Torunczyk's theorem [2] on the homeomorphism of all (nonseparable) Banach spaces of a given dimension. The existence of a linear injection is equivalent to the existence in Y of a countably total set of linear functionals (in other words, to the weak* separability of the conjugate space Y^*). We note that, according to [3], the existence of a resolving linear regularizer (in the case of a separable X) implies the weak* separability of Y^*, i.e. in this case the existence of a linear injection is not only a sufficient but also a necessary condition for the existence of a resolving regularizer. We also note that if Y^* is weak* separable and the linear operator $A: X \to Y$ admits a linear regularizer $(R_n)_1^\infty$, then a resolving regularizer can be defined by the following formulas:

$$\overline{R}_{2n-1}y = R_n y, \quad \overline{R}_{2n}y = R_n y + \sum_{k=1}^{n} \lambda_k g_k (AR_n y - y) x_k,$$

where $(x_k)_1^\infty$ is a minimal sequence in X, $(g_k)_1^\infty$ is a countable total subset of Y^*, and $(\lambda_k)_1^\infty$ is a sequence of positive coefficients, converging sufficiently fast to zero.

The concept of a resolving regularizer has a deficiency: a subsequence of a resolving regularizer need not be a resolving regularizer. In connection with this we introduce the following definition, also due to Domanskii. A sequence $(R_n)_1^\infty$ is said to be a strictly resolving regularizer if each of its subsequences is a resolving regularizer.

Now we consider the elucidation of the following question: under what conditions on A, X, and Y does there exist a strictly resolving regularizer?

We need some facts from the theory of Banach spaces. A sequence $(x_n)_1^\infty$ of elements of a space X is said to be minimal if it admits a conjugate system $(f_n)_1^\infty$ (i.e., such that $f_i(x_j) = \delta_{ij}$, i, j = 1, 2, ...). The subspace spanned by the elements $(x_i)_1^n$ will be denoted by X_n. To each element $x \in X$ one associates the (in general, divergent) series $\sum_1^\infty f_n(x) x_n$. Its partial sums are denoted by $S_n x$, so that S_n is a linear continuous operator, acting from X into X_n. We say that the system $(x_i, f_i)_1^\infty$ is a normalizing M-basis ("M" from Markushevich) if $(x_n)_1^\infty$ is a complete set in X and the linear hull of the set $(f_n)_1^\infty$ is a normalizing subspace in X^*. A sequence $\Gamma \subset X^*$ is said to be normalizing if $\sup\{f(x): f \in \Gamma, \|f\| \le 1\} \ge \gamma\|x\|$ for some $\gamma > 0$ and for all $x \in X$. In [4], it is proved that each normalizing M-basis has the following "restoring" property relative to the biorthogonal expansions $x \sim \Sigma f_n(x)x_n$: there exists a sequence of continuous (in general, nonlinear) operators T_n: $X_n \to X$ such that for all $x \in X$ one has

$$\lim_{n\to\infty} T_n S_n x = x, \tag{4}$$

while for each collection of numerical coefficients $(a_i)_1^\infty$ one has

$$f_j\left(T_n\left(\sum_{i=1}^{n} a_i x_i\right)\right) = a_j, \quad j = 1, 2, \ldots, n. \tag{5}$$

This property of normalizing M-bases will be used for the proof of Theorem 2. We also note that, according to [5], a linear injective operator A, acting from a separable space X into an arbitrary space Y, admits a regularizer (not necessarily linear) if and only if A^*Y^* is a normalizing subspace in X.

<u>LEMMA</u>. Let X be separable and infinite-dimensional, let Y^* be weak * separable, let A: $X \to Y$ be a linear injective operator, and assume that A^*X^* is a normalizing subspace in X^*. Then in X there exists a normalizing M-basis $(x_i, f_i)_1^\infty$ such that $f_i = A^*g_i$ and $(g_i)_1^\infty$ is a total set of linear functionals over Y.

Proof. We select in Y^* a total sequence $(h_i)_1^\infty$, lying outside $\ker A^*$. We complete it by elements $(b_i)_1^\infty \subset Y^* \setminus \ker A^*$ so that the set $(h_n)_1^\infty \cup (b_n)_1^\infty$ remains total and the linear hull of the set $(A^*h_n)_1^\infty \cup (A^*b_n)_1^\infty$ is a normalizing subspace of Y^*. This can be done on the basis of the conditions of the lemma. Now, according to [6, p. 224, Theorem 8.1], we can form an M-basis $(x_i, f_i)_1^\infty$ such that $f_i = A^*g_i$, the linear hull $\mathrm{Lin}(g_i)_1^\infty$ coincides with $\mathrm{Lin}\{(h_i)_1^\infty \cup (b_i)_1^\infty\}$ and, consequently, $\mathrm{lin}(A^*g_i)_1^\infty$ is a normalizing set.

THEOREM 2. If X is infinite-dimensional and separable, Y^* is weak * separable, while the linear injection $A: X \to Y$ admits a regularizer, then A admits also a strictly resolving regularizer.

Proof. According to the above given facts, in X there exist a normalizing M-basis $(x_i, f_i)_1^\infty$ and a sequence of continuous operators $T_n = X_n \to X$, satisfying conditions (4) and (5) and, moreover, $f_i = A(h_i$ and $(g_i)_1^\infty$ is total. We define the desired resolving regularizer in the following manner:

$$R_n y = T_n\left(\sum_{i=1}^{n} g_i(y) x_i\right), \quad y \in Y, \quad n = 1, 2, \ldots. \tag{6}$$

We verify that $(R_n)_1^\infty$ is a regularizer. Indeed,

$$R_n A x = T_n\left(\sum_{i=1}^{n} g_i(Ax) x_i\right) = T_n\left(\sum_{i=1}^{n} f_i(x) x_i\right) \to x.$$

Assume now that $R_n y \to x$ for $n \to \infty$. According to (5) we have

$$f_j(R_n y) = f_j\left(\sum_{i=1}^{n} g_i(y) x_i\right) = g_j(y), \quad 1 \leqslant j \leqslant n.$$

On the other hand,

$$\lim_{n\to\infty} f_j(R_n y) = f_j(x) = g_j(Ax), \quad j = 1, 2, \ldots.$$

Comparing the last two equalities, by virtue of the fact that the set $(g_j)_1^\infty$ is total, we obtain that $y = Ax$. Thus, $(R_n)_1^\infty$ is a resolving regularizer. The reasoning remains valid if we restrict ourselves to the consideration of any subsequence of indices $(n_k)_1^\infty$. Thus, $(R_n)_1^\infty$ is a strictly resolving regularizer.

Finally, we proceed to the consideration of finite-dimensional linear regularizers.

THEOREM 3. Let X be separable and infinite-dimensional, let Y^* be weak * separable, and let $A: X \to Y$ be a linear injection, admitting a regularizer $(R_n)_1^\infty$, composed of finite-dimensional linear operators. Then A admits also a strictly resolving finite-dimensional linear regularizer.

Proof. We define the desired regularizer by the formula

$$\bar{R}_n y = R_n y + \sum_{k=1}^{n} \lambda_k g_k(AR_n y - y) e_{n+k}, \quad n = 1, 2, \ldots, \tag{7}$$

where $(g_k)_1^\infty$ is a normalized total sequence of linear functionals from Y^*, $(e_n)_1^\infty$ is a normalized sequence selected in a special manner from X, $(\lambda_n)_1^\infty$ is a positive numerical sequence, guaranteeing the inequality

$$\left\| \sum_{k=1}^{n} \lambda_k g_k(y) e_{n+k} \right\| \leqslant \|y\| \text{ for all } \quad y \in Y. \tag{8}$$

We select the sequence $(e_n)_1^\infty$ in the following manner. We fix ε, $0 < \varepsilon < 1$. On the unit sphere of the finite-dimensional subspace $R_1 Y \subset X$ we select a finite ε-net and for each element of this ε-net we select a supporting linear functional [we recall that a normalized linear functional f X^* is said to be supporting with respect to the element $x \in X$ if $f(x) = \|x\|$]. The intersection of the kernels of these functionals is a subspace of finite codimension $X^1 \subset X$ which is "ε-orthogonal" to R_1Y in the sense that for all $x \in R_1Y$ and $z \in X^1$ we have $\|x + z\| \geq (1 - \varepsilon)\|x\|$. As e_1 we select any normalized element from X^1. We consider the linear span of the subspaces R_1Y, R_2Y and the element e_1; on the unit sphere of the finite-dimensional subspace $\mathrm{Lin}(R_1Y, R_2Y, e_1)$ we select a finite ε-net, including in it also the ε-net selected at the preceding step; for each element of the ε-net we select a supporting functional. The intersection of the kernels of these functionals is a subspace $X^2 \subset X^1 \subset X$ of finite codimension, which is ε-orthogonal to the subspace $\mathrm{Lin}(R_1Y, R_2Y, e_1)$. For e_2 we select an arbitrary normalized element of X^2. We continue this process indefinitely and we obtain the required sequence $(e_n)_1^\infty$. We mention those of its properties that are needed: for each x R_nY and each finite collection of coefficients $(c_i)_1^N$ we have

$$\left\|x + \sum_{i=1}^{N} c_i e_{n+i}\right\| \geqslant (1 - \varepsilon)\|x\|, \quad n = 1, 2, \ldots, \tag{9}$$

for each collection of coefficients $(a_i)_1^n$ we have

$$\left\|\sum_{i=1}^{n} a_i e_i\right\| \geqslant \frac{1-\varepsilon}{2} \max\{|a_i|:\ 1 \leqslant i \leqslant n\}. \tag{10}$$

We verify that the sequence defined by formula (7) is a regularizer. According to relations (7) and (8), we have

$$\|\bar{R}_n Ax - R_n Ax\| = \left\|\sum_{k=1}^{n} \lambda_k g_k (A(R_n Ax - x)) e_{n+k}\right\| \leqslant \|A\| \|R_n Ax - x\|.$$

Since $\lim R_nAx = x$, we also have $\lim \bar{R}_nAx = x$. Now we prove that $(\bar{R}_n)_1^\infty$ is a strictly resolving regularizer. Assume that n runs through some subsequence of the natural series and let $\lim_n \bar{R}_n y = x$. If we denote $y - Ax = u$, then $\bar{R}_n u \to 0$. In view of the inequality (9) we have $\|\bar{R}_n u\| \geq (1 - \varepsilon)\|R_n u\|$, so that $R_n u \to 0$ for the same sequence of indices. Thus, also, the last term in equality (7) tends to zero:

$$\lim_n \left\|\sum_{k=1}^{n} \lambda_k g_k (AR_n u - u) e_{n+k}\right\| = 0.$$

According to (10), for any natural number m we have

$$\lim_n |\lambda_m g_m (AR_n u - u)| = 0,$$

from where $g_m(u) = 0$ for all m. Consequently, by virtue of the fact that the set $(g_n)_1^\infty$ is total, we have $u = 0$, i.e., $y = Ax$.

We note that, under the assumptions of Theorems 1 or 3, the existence of a linear finite-dimensional resolving regularizer has been established also in [7].

At the consideration of the question of the regularizability of an operator $A: X \to Y$ there arise three different problems: on the existence of 1) a regularizer, 2) a linear regularizer, and 3) a linear finite-dimensional regularizer. As one can see from Theorem 1, in the case of a separable space X, all three problems (under the natural assumption of the weak separability of the space Y^*) are equivalent to the analogous problems for the resolving regularizer. Theorems 2 and 3 show that for strictly resolving regularizers (if X is separable) the first and third problems can be solved in the case when they can be solved for resolving regularizers. The question regarding the second problem remains open:

Question. Assume that it is known that the operator A, acting from a separable space X, admits a linear resolving regularizer. Does A admit a strictly resolving regularizer?

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THERE IS NO LOCAL UNCONDITIONAL STRUCTURE IN ANISOTROPIC SPACES OF SMOOTH FUNCTIONS

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INTRODUCTION

For every finite subset A of an integer lattice $\mathbf{Z}_+^n\,(=(\mathbf{Z}_+)^n)$ there corresponds a Banach space $C^A(\mathbf{T}^n)$ consisting of all the continuous functions f on the torus $\mathbf{T}^n$ whose derivatives (in the sense of the theory of generalized functions) corresponding to the multiindices from A are also continuous functions. The norm in $C^A(\mathbf{T}^n)$ is specified by the equation

$$\|f\|_{C^A}=\|f\|_\infty+\sum_{a\in A}\|D^af\|_\infty,$$

where $\|\cdot\|_\infty$ is the standard supremum norm, $D^a=\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$ for $a = (\alpha_1, \dots, \alpha_n)$ and $(\partial_j f)(z_1, \dots, z_n)\overset{\text{def}}{=}\frac{\partial}{\partial\theta}f(z_1, \dots, z_{j-1}, e^{i\theta}, z_{j+1}, \dots, z_n)|_{\theta=t_j}$, whenever $z_j = e^{it_j}$.

When $A=\{a=(\alpha_1, \dots, \alpha_n)\in\mathbf{Z}_+^n\colon |a|=\alpha_1+\dots+\alpha_n\leqslant l\}$ the space $C^A(\mathbf{T}^n)$ is nothing else but the standard space $C^{(\ell)}(\mathbf{T}^n)$ of all ℓ-fold continuously differentiable functions. For n = 1 this space is isomorphic to the space $C(\mathbf{T})$ for any ℓ (i.e., linearly homeomorphic). (Indeed, the operator $\partial_1{}^\ell$ maps $C^{(\ell)}(\mathbf{T})$ linearly onto $C(\mathbf{T})$ and its kernel contains only constants.) For $n \geq 2$ the situation substantially differs.

Grothendieck [1] was first to announce that for $n \geq 2$ and $\ell \geq 1$ the space $C^{(\ell)}(\mathbf{T}^n)$ is not isomorhic to the direct factor of C(K) for any compact K. The first proof of this fact was given by Khenkin [2]. Later Kislyakov [3] showed that for such n and ℓ the space $C^{(\ell)}(\mathbf{T}^n)$ is not isomorphic to any quotient space of the space C(K). Then Kislyakov [4] and Kwapien and Pelczynski [5] proved, by use of different techniques, that the space $C^{(\ell)}(\mathbf{T}^n)$ with $n \geq 2$ and $\ell \geq 1$ has no local unconditional structure. Roughly speaking, this latter fact means that "typical" finite-dimensional subspaces of $C^{(\ell)}(\mathbf{T}^n)$ are "far away" from the subspaces of a space with a 1-unconditional basis admitting "small norm" projection. Following [6], we now give a definition of the local unconditional structure.

A Banach space X is said to have a local unconditional structure if there exists a constant C > 0 such that for any finite-dimensional subspace F in X there are a Banach space E

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