Mathematical Notes, Vol. 61, No. 5, 1997

Continuation of a Linear Operator to an Involution Operator

M. I. Kadets and K. É. Kaibkhanov UDC 517

ABSTRACT. A bounded linear operator $A: X \to X$ in a linear topological space X is called a *p-involution operator*, $p \geq 2$, if $A^p = I$, where I is the identity operator. In this paper, we describe linear p-involution operators in a linear topological space over the field $\mathbb{C}$ and prove that linear operators can be continued to involution operators.

KEY WORDS: linear topological space, linear operator, p-involution operator.

The main goal of this paper is to prove that a linear operator can be continued to an involution operator.

A linear bounded operator $A: X \to X$ in a linear topological space X is called a *p-involution operator*, $p \geq 2$, if $A^p = I$, where I is the identity operator; for $p = 2$, A is called an *involution*. In this paper, consider only bounded linear operators. The operators of the form $A = e^{2\pi ki/p}I$, $0 \leq k \leq p-1$, provide the simplest example of linear p-involution operators in a linear topological space X over $\mathbb{C}$. The following statement shows that the set of such operators is essentially exhausted by the operators of the form $A = e^{2\pi ki/p}I$, $0 \leq k \leq p-1$.

Theorem 1. *Suppose that $A: X \to X$ is a linear p-involution operator in a linear topological space X over $\mathbb{C}$. Let $\varepsilon = e^{2\pi i/p}$. Then there exist subspaces $X_1, X_2, \dots, X_p \subset X$ such that*

1) $X = X_1 \oplus X_2 \oplus \cdots \oplus X_p$;
2) $A|_{X_k} = \varepsilon^k I|_{X_k}$, $0 \leq k \leq p-1$.

Proof. Consider the operators

$$Q_k = \frac{1}{p}\left(\sum_{j=0}^{p-1}(\varepsilon^{p-k}A)^j\right), \qquad 1 \leq k \leq p;$$

here we assume $A^0 = I$. Straightforward verification shows that the Q_k are projections (that is, $Q_k^2 = Q_k$). We set $X_k = Q_k(X)$. One can easily see that the X_k have the desired properties. □

Note that for $p = 2$, this theorem also holds for spaces over the field $\mathbb{R}$.

Theorem 2. *Let $A: X \to X$ be a linear operator in a linear topological space X (over the field $\mathbb{R}$ or $\mathbb{C}$) and let $p \geq 2$. Then there exists a linear topological space $Z \supset X$ with the projection operator $P: Z \to X$ (X can be complemented to Z) and a linear p-involution operator $B: Z \to Z$ such that $PB|_X = A$.*

Proof. Let $X_2, X_3, \dots, X_p$ be copies of the space X, $X_1 = X$, and let $U_k: X \to X_k$ be the canonical isometries, $U_1 = I|_X$. We form the direct sum $Z = X_1 \oplus X_2 \oplus \cdots \oplus X_p$. Let $Q_k: Z \to X_k$ be the canonical projections. We define the operators

$$V_1 = U_2U_1^{-1}Q_1, \quad V_2 = U_3U_2^{-1}Q_2, \quad \dots, \quad V_{p-1} = U_pU_{p-1}^{-1}Q_{p-1}, \quad V_p = U_1U_p^{-1}Q_p,$$
$$V = V_1 + V_2 + \cdots + V_p.$$

Next, we define an operator $S: Z \to X$ by setting $S = U_2^{-1}Q_2 + U_3^{-1}Q_3 + \cdots + U_p^{-1}Q_p$ and construct the isomorphism $T = I + AS$. One can readily see that $T^{-1} = I - AS$. Finally, we define the operator $B: Z \to Z$ with the required properties by setting $B = TVT^{-1}$ (in this case, $P = Q_1$). □

Translated from *Matematicheskie Zametki*, Vol. 61, No. 5, pp. 671–676, May, 1997.
Original article submitted July 13, 1995.

0001-4346/97/6156-0561 $18.00

Let us consider the case of a Hilbert space. It is natural to ask how to choose the best (with respect to the norm) operator B in Theorem 2. The previous proof implies the estimate $\|B\| \le (1+\|A\|)^2$ for $p=2$. The following statement shows that this estimate can be improved.

Theorem 3. *For any linear operator $A\colon H \to H$ in a Hilbert space H (over the field $\mathbb{R}$ or $\mathbb{C}$), there exists a Hilbert space $E \supset H$ and a linear involution $B\colon E \to E$ such that*

1) *$PB|_H = A$, where $P\colon E \to H$ is the operator of orthogonal projection;*
2) *the following estimate holds:*

$$\|B\| \le \begin{cases} \sqrt{\dfrac{17+4\sqrt{2}}{2}} & \text{if } \|A\| \le \sqrt{2}; \\ \sqrt{4\|A\|^2 + \dfrac{1}{\|A\|^2} + 2\sqrt{2}} & \text{if } \|A\| > \sqrt{2}. \end{cases}$$

We prove this theorem in the case of an infinite-dimensional separable Hilbert space over $\mathbb{R}$; this proof, however, remains valid in other cases. To prove Theorem 3, we need some auxiliary statements.

We introduce the following notation: $\langle x, y\rangle$ is the inner product of elements x and y; $[x_j]_{j=1}^{\infty}$ is the closed linear span of the vectors $\{x_j\}_{j=1}^{\infty}$.

Let $A\colon H \to H$ be a linear operator in a Hilbert space H, and let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in H; we write $g_j = Ae_j$. Let F be a Hilbert space with an orthonormal basis $\{f_j\}_{j=1}^{\infty}$. We form the Hilbert space $Z = (H \oplus F)_2$ with the natural inner product ($\langle e_j, f_k\rangle = 0$) and suppose that $Q\colon Z \to H$ is the natural projection operator. If $h \in Z$, then we sometimes write $h = (e, f)$, having in mind that $e \in H$ and $f \in F$. We set $h_j = (g_j, tf_j) \in (H \oplus F)_2$, where $t > 0$ (later, we choose an appropriate t), and write $[h_j]_{j=1}^{\infty} = Y$.

Lemma 1. *$\{h_j\}_{j=1}^{\infty}$ is a basis in Y equivalent to the basis $\{e_j\}_{j=1}^{\infty}$ in H.*

Proof. For an arbitrary system of numbers $\{a_j\}_{j=1}^{\infty}$, we have

$$t^2\Big(\sum_{j=1}^{\infty} a_j^2\Big) = \sum_{j=1}^{\infty} t^2 a_j^2 \le \Big\|\sum_{j=1}^{\infty} a_j g_j\Big\|^2 + \sum_{j=1}^{\infty} t^2 a_j^2 = \Big\|\sum_{j=1}^{\infty} a_j h_j\Big\|^2 = \Big\|\Big(\sum_{j=1}^{\infty} a_j A e_j, \sum_{j=1}^{\infty} a_j t f_j\Big)\Big\|^2$$
$$= \Big\|A\Big(\sum_{j=1}^{\infty} a_j e_j\Big)\Big\|^2 + t^2 \sum_{j=1}^{\infty} a_j^2 \le \|A\|^2 \sum_{j=1}^{\infty} a_j^2 + t^2 \sum_{j=1}^{\infty} a_j^2 = (\|A\|^2 + t^2)\sum_{j=1}^{\infty} a_j^2. \quad \square$$

Lemma 2. *For any $e \in H$ and $h \in Y$,*

$$|\langle e, h\rangle| \le \frac{\|A\| \cdot \|e\| \cdot \|h\|}{\sqrt{\|A\|^2 + t^2}}. \tag{1}$$

Proof. By setting $h = (g, f) \in (H \oplus F)_2$, $g = \sum_{j=1}^{\infty} a_j g_j$, we obtain

$$|\langle e, h\rangle| = |\langle e, g\rangle| \le \|e\| \cdot \|g\| = \frac{\|e\| \cdot \|g\| \cdot \|h\|}{\|h\|} = \frac{\|e\| \cdot \big\|\sum_{j=1}^{\infty} a_j g_j\big\| \cdot \|h\|}{\sqrt{\big\|\sum_{j=1}^{\infty} a_j g_j\big\|^2 + \sum_{j=1}^{\infty} t^2 a_j^2}} \le \frac{\|A\| \cdot \|e\| \cdot \|h\|}{\sqrt{\|A\|^2 + t^2}}. \quad \square$$

Lemma 3. *For any $\{a_j\}_{j=1}^{\infty} \subset \mathbb{R}$ and any $h \in Y$,*

$$\Big\|\sum_{j=1}^{\infty} a_j e_j + h\Big\| \ge \frac{\big\|\sum_{j=1}^{\infty} a_j e_j\big\| t}{\sqrt{\|A\|^2 + t^2}}. \tag{2}$$

Proof. For brevity, we write $\sum_{j=1}^{\infty} a_j e_j = e$. Then it follows from (1) that

$$\|e+h\|^2 = \|e\|^2 + 2\langle e, h\rangle + \|h\|^2 = \|e\|^2 \left[\left(\frac{\|h\|}{\|e\|} - \frac{\|A\|}{\sqrt{\|A\|^2 + t^2}} \right)^2 + \frac{t^2}{\|A\|^2 + t^2} \right] \geq \frac{\|e\|^2 t^2}{\|A\|^2 + t^2},$$

and the lemma is thereby proved. □

Lemma 4. *After the natural renumbering* $e_1, h_1, e_2, h_2, \dots$ *, the set* $\{e_j\}_{j=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$ *forms a basis in* Z.

Proof. Assume that $x \in Z$ is arbitrary and

$$x = \sum_{j=1}^{\infty} a_j e_j + \sum_{k=1}^{\infty} b_k f_k.$$

Then

$$x = \sum_{j=1}^{\infty} a_j e_j + \frac{1}{t}\sum_{k=1}^{\infty} b_k h_k - \frac{1}{t}\sum_{k=1}^{\infty} b_k g_k = \left(\sum_{j=1}^{\infty} a_j e_j - \frac{1}{t}\sum_{k=1}^{\infty} b_k g_k \right) + \frac{1}{t}\sum_{k=1}^{\infty} b_k h_k.$$

Since

$$\frac{1}{t}\sum_{k=1}^{\infty} b_k g_k \in H,$$

it follows that there exist numbers $\{d_j\}_{j=1}^{\infty}$ such that

$$\frac{1}{t}\sum_{k=1}^{\infty} b_k g_k = \sum_{j=1}^{\infty} d_j e_j.$$

Hence,

$$x = \sum_{j=1}^{\infty} (a_j - d_j) e_j + \frac{1}{t}\sum_{k=1}^{\infty} b_k h_k,$$

that is, x can be decomposed with respect to the elements $\{e_j\}_{j=1}^{\infty} \cup \{h_k\}_{k=1}^{\infty}$, and the coefficients in this decomposition are determined uniquely. It remains to prove that there exists a $C > 0$ such that the inequality [1, Proposition 1.a.3]

$$\left\| \sum_{j=1}^{n+m} a_j e_j + \sum_{k=1}^{l+s} b_k h_k \right\| \geq C \left\| \sum_{j=1}^{n} a_j e_j + \sum_{k=1}^{l} b_k h_k \right\|.$$

is satisfied for any n, m, l, and $s \in \mathbb{N}$. Obviously, it suffices to prove this inequality for

$$\left\| \sum_{j=1}^{n} a_j e_j + \sum_{k=1}^{l} b_k h_k \right\| = 1.$$

For brevity, we write

$$e = \sum_{j=1}^{n} a_j e_j, \qquad h = \sum_{k=1}^{l} b_k h_k.$$

Let $\|e+h\| = 1$. Then we have at least one of the cases: 1) $\|e\| \geq 1/2$; 2) $\|h\| \geq 1/2$. Let us consider each of them.

1) $\|e\| \geq 1/2$. Using (2), we obtain

$$\left\|\sum_{j=1}^{n+m} a_j e_j + \sum_{k=1}^{l+s} b_k h_k\right\| \geq \frac{\|\sum_{j=1}^{n+m} a_j e_j\| t}{\sqrt{\|A\|^2 + t^2}} \geq \frac{\|\sum_{j=1}^{n} a_j e_j\| t}{\sqrt{\|A\|^2 + t^2}} \geq \frac{t}{2\sqrt{\|A\|^2 + t^2}}.$$

2) $\|h\| \geq 1/2$. We have

$$\|h\| = \left\|\sum_{k=1}^{l} b_k h_k\right\| = \left\|\left(\sum_{k=1}^{l} b_k g_k, t\sum_{k=1}^{l} b_k f_k\right)\right\| = \sqrt{\left\|\sum_{k=1}^{l} b_k g_k\right\|^2 + t^2 \sum_{k=1}^{l} b_k^2} \geq \frac{1}{2}.$$

Since $\|g_k\| \leq \|A\|$, the previous inequality implies

$$\sqrt{\|A\|^2 + t^2} \cdot \sqrt{\sum_{k=1}^{l} b_k^2} \geq \frac{1}{2},$$

and hence,

$$\sqrt{\sum_{k=1}^{l} b_k^2} \geq \frac{1}{2\sqrt{\|A\|^2 + t^2}}.$$

Therefore,

$$\left\|\sum_{j=1}^{n+m} a_j e_j + \sum_{k=1}^{l+s} b_k h_k\right\| \geq t\sqrt{\sum_{k=1}^{l+s} b_k^2} \geq t\sqrt{\sum_{k=1}^{l} b_k^2} \geq \frac{t}{2\sqrt{\|A\|^2 + t^2}}.$$

Thus, in both cases we can take $C = t/\left(2\sqrt{\|A\|^2 + t^2}\right)$. □

Proof of Theorem 3. 1) We set $E = Z$. By Lemma 4, we have $E = Z = H \oplus Y$. We define an operator $B : E \to E$ as follows:

$$B\left(\sum_{j=1}^{\infty} a_j e_j + \sum_{k=1}^{\infty} b_k h_k\right) = \sum_{k=1}^{\infty} b_k e_k + \sum_{j=1}^{\infty} a_j h_j.$$

Since the bases $\{e_j\}_{j=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ are equivalent, the operator B is well defined. We set $P = Q$. For an arbitrary

$$x = \sum_{j=1}^{\infty} a_j e_j \in H,$$

we have

$$PBx = PB\left(\sum_{j=1}^{\infty} a_j e_j\right) = P\left(\sum_{j=1}^{\infty} a_j h_j\right) = P\left(\left(\sum_{j=1}^{\infty} a_j g_j, \sum_{j=1}^{\infty} a_j t f_j\right)\right)$$
$$= \sum_{j=1}^{\infty} a_j g_j = \sum_{j=1}^{\infty} a_j A e_j = A\left(\sum_{j=1}^{\infty} a_j e_j\right) = Ax,$$

that is, $PB|_H = A$, and statement 1) in Theorem 3 is thereby proved.

2) Let us estimate $\|B\|$. Let

$$\left\|\sum_{j=1}^{\infty} a_j e_j + \sum_{k=1}^{\infty} b_k h_k\right\| = 1.$$

Then it follows from (2) that

$$\sum_{j=1}^{\infty} a_j^2 = \left\| \sum_{j=1}^{\infty} a_j e_j \right\|^2 \le \frac{\|A\|^2 + t^2}{t^2}. \tag{3}$$

We have

$$\sqrt{t^2 \sum_{j=1}^{\infty} b_j^2} \le \left\| \sum_{j=1}^{\infty} a_j e_j + \sum_{k=1}^{\infty} b_k h_k \right\| = 1.$$

Hence,

$$\sum_{k=1}^{\infty} b_k^2 \le \frac{1}{t^2}. \tag{4}$$

Using (3) and (4), we obtain

$$\begin{aligned}
\left\| B\Big(\sum_{j=1}^{\infty} a_j e_j + \sum_{k=1}^{\infty} b_k h_k\Big) \right\|^2 &= \left\| \sum_{k=1}^{\infty} b_k e_k + \sum_{j=1}^{\infty} a_j h_j \right\|^2 = \left\| \sum_{k=1}^{\infty} b_k e_k + \sum_{j=1}^{\infty} a_j g_j \right\|^2 + t^2 \sum_{j=1}^{\infty} a_j^2 \\
&\le \left(\left\| \sum_{k=1}^{\infty} b_k e_k \right\| + \left\| \sum_{j=1}^{\infty} a_j g_j \right\| \right)^2 + t^2 \sum_{j=1}^{\infty} a_j^2 \le \left(\sqrt{\sum_{k=1}^{\infty} b_k^2} + \|A\| \sqrt{\sum_{j=1}^{\infty} a_j^2} \right)^2 + \|A\|^2 + t^2 \\
&= \sum_{k=1}^{\infty} b_k^2 + 2\|A\| \sqrt{\sum_{k=1}^{\infty} b_k^2} \cdot \sqrt{\sum_{j=1}^{\infty} a_j^2} + \|A\|^2 \sum_{j=1}^{\infty} a_j^2 + \|A\|^2 + t^2 \\
&\le \frac{1}{t^2} + \frac{2\|A\| \sqrt{\|A\|^2 + t^2}}{t^2} + \frac{\|A\|^2 (\|A\|^2 + t^2)}{t^2} + \|A\|^2 + t^2.
\end{aligned}$$

Thus, $\|B\|^2$ cannot exceed the last term in this chain. We consider the two cases: a) $\|A\| \le \sqrt{2}$ and b) $\|A\| > \sqrt{2}$.

a) $\|A\| \le \sqrt{2}$. We set $t = \sqrt{2}$. Then we obtain

$$\|B\| \le \sqrt{\frac{1}{2} + 2\sqrt{2} + 8} = \sqrt{\frac{17 + 4\sqrt{2}}{2}}.$$

b) $\|A\| > \sqrt{2}$. In this case we set $t = 1$. Then one readily verify that

$$\|B\| \le \sqrt{4\|A\|^2 + \frac{1}{\|A\|^2} + 2\sqrt{2}}.$$

Theorem 3 is proved. □

References

1. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I. Sequence Spaces*, Springer, Berlin (1977).

(M. I. Kadets) Khar'kov State Academy of Municipal Economy

(K. É. Kaibkhanov) Taganrog Radiotechnical University

Translated by M. A. Shishkova