1) tetrahedron, then Γ is a four-sectioned polygonal arc, determined by six parameters within the given polyhedron,

2) cube or 3) octahedron, then Γ is a six-sectioned polygonal arc, determined by twelve parameters,

4) icosahedron or 5) dodecahedron, then Γ is a ten-sectioned polygonal arc, determined by thirty parameters.

One can see the existence of the closed path Γ directly (cf. the figure). In cases 1), 2), and 3), the path Γ completely determines the polyhedron, since either in the sequence of faces $\{\Pi_k\}$ all faces of the polyhedron occur, or in the path Γ all its vertices occur. In cases 4) and 5) this is not so. In case 4) the path Γ does not contain two points which can be taken arbitrarily. The arbitrariness in determining one point in relation to Γ is determined by three parameters: the dihedral angle, the angle between two edges, and the length of an edge. Hence, the arbitrariness in defining a polyhedron with net of edges of an icosahedron consists of arbitrarily giving a ten-sectioned polygonal arc + six parameters, i.e., thirty parameters. One can make analogous considerations in case 5).

Since the angles ψ_i between edges of the polyhedron are supplementary to the curvature angles φ_i of the path Γ, and the angles between faces θ_i are the torsion angles of this polygonal arc, writing the closedness conditions for the curve Γ, we get conditions on the parameters defining the polyhedron itself.

A NORMABILITY CONDITION FOR FRECHET SPACES

V. M. Kadets and M. I. Kadets

Introduction. Several papers have been published recently, devoted to the generalization of James' known theorem on the reflexivity of a Banach space to linear metrizable spaces (see references cited in [1]). Zarnadze [1] has formulated, among other results in this direction, an interesting statement connecting the normability of a Frechet space with the presence in it of some continuous norm. Unfortunately, his proof is not valid.

The purpose of the present paper is to prove Zarnadze's statement, at the same time setting it free of redundant assumptions (Theorem 1), to prove a statement that is dual in a certain sense (Theorem 2), and to give examples which show that Theorems 1 and 2 cannot be extended to complete, nonmetrizable, locally convex spaces.

1. Definitions and Auxiliary Results

LEMMA 1 [2]. Let X be a complete metric space, let Y be a metric space, and let u be a continuous mapping of X into Y. Assume further that for each $r > 0$ there exists $\rho > 0$ such that for any $x \in X$ the image of the closed ball $B_r(x)$ of radius r and center in x is dense in the ball $B_\rho(u(x))$. Then for any $a > r$ we have the inclusion $(B_a(x)) \supset B_\rho(u(x))$.

Definition 1. A subset R of the unit sphere S(E) of a reflexive Banach space E is a boundary if for each linear functional $f \in E'$ there exists $x \in R$ such that $f(x) = \|f\|$.

We note that by virtue of James' well-known theorem (see [3]) on linear functionals which attain their norms, the concept of a boundary can be introduced only for reflexive spaces.

LEMMA 2 [4]. If a boundary R of a reflexive Banach space E is covered by an increasing sequence of absolutely convex closed sets, then at least one of them has a nonempty interior.

LEMMA 3. Let X be a Frechet space and let Y be a reflexive Banach space with boundary R. Let T be a continuous linear operator from X into Y and let $T(X) \supset R$. Then T is a surjection.

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Proof. Assume that on X there is given a metric defining the topology and let B_r be a ball of radius $r > 0$ with center at zero. Let $V \subset B_r$ be a convex neighborhood of zero in X_∞. Then $X = \bigcup_{n=1}^{\infty} nV$. Since by the assumption of the lemma we have $T(X) \supset R$, it follows that $\bigcup_{n=1}^{\infty} T(nV) \supset R$. Consequently, by Lemma 2, the closure of the set T(V) has a nonempty interior and thus the set $T(B_r)$ is dense in some ball of the space Y. Since r has been selected in an arbitrary manner, it follows by Lemma 1 that the set $T(B_r)$ contains some ball of the space Y, i.e., T is a surjection.

In the case when X is a Banach space, this proposition has been proved in [5, Theorem 1].

Definition 2. Let V be a closed absolutely convex subset of a locally convex space X; let $p(\cdot)$ be the seminorm generated by the set V in its linear hull Lin V. A linear functional $f \in X'$ will be said to be attaining for the point $x \in \mathrm{Lin}\, V$ (and denoted by f_x) if $f_x(x) = p(x)$ and $f_x(y) \leq p(y)$ for all $y \in \mathrm{Lin}\, V$. By the algebraic boundary ∂V of a set V we mean the set of those $x \in V$ for which $p(x) = 1$. A point $x \in \partial V$ will be said to be attainable if there exists for it an attaining functional f_x. If the set V has a nonempty interior, then, by the Hahn–Banach theorem, each point of its boundary ∂V is attainable.

2. Fundamental Results

THEOREM 1. Let X be a Frechet space and let $p(\cdot)$ be a continuous norm on X, satisfying the following condition (J): each linear functional $f \in X'$, bounded on the set

$$U_p = \{x \in X : p(x) \leqslant 1\},$$

attains on it its supremum. Then $p(\cdot)$ generates the initial topology on X which, consequently, turns out to be isomorphic to a reflexive Banach space.

Proof. We consider the space Y, the completion of the space X with respect to the norm $p(\cdot)$. We denote by T the identity imbedding of X into Y. Since the norm $p(\cdot)$ is continuous, it follows that T is a continuous injection. Since $p(\cdot)$ satisfies condition (J), by James' theorem [3], Y is reflexive and T(X) contains some boundary in the space Y. Consequently, by Lemma 3, the operator T is a surjection. Then, according to Banach's inverse operator theorem, T is an isomorphism between the Frechet space X and the reflexive Banach space Y. The theorem is proved.

We proceed to the proof of the dual statement which, in some sense, is the inverse of the Hahn–Banach theorem.

THEOREM 2. Let X be a Frechet space; let V be a closed, bounded, absolutely convex set, whose linear hull is dense in X. Assume that V satisfies the following condition (J*): each point of its algebraic boundary is attainable. Then V is a neighborhood of zero in X, defining the initial topology on X which, consequently, turns out to be isomorphic to a Banach space.

Proof. Let $p_1(\cdot)$, $p_2(\cdot)$, ... be an increasing sequence of seminorms, defining the topology of the space X. We denote by $p(\cdot)$ the seminorm generated by the set V on Lin V. We assume that V has an empty interior (i.e., it is not a neighborhood). This means that, for any of the seminorms, defining jointly the topology, we have $\inf\{p_n(x) : x \in \partial V\} = 0$. We select an element $e_1 \in \partial V$ such that $p_1(e_1) \leq 4^{-1}$. We select the coefficient $a_1 = (1 + 2^{-1})^{-1}$; then $p(a_1 e_1) = (1 + 2^{-1})^{-1}$. We consider the closed subspace of codimension one $X_1 = \ker f_{e_1}$. Clearly, $\mathrm{Lin}\, V \cap X_1$ is dense in X_1 and $V \cap X_1$ has an empty interior. Therefore, one can select $e_2 \in \partial V$ such that $f_{e_1}(e_2) = 0$ and $p_2(e_2) \leq 4^{-2}$. According to the triangle inequality, we have

$$\max\{p(a_1 e_1 + e_2), p(a_1 e_1 - e_2)\} \geqslant 1.$$

Since $p(a_1 e_1) = (1 + 2^{-1})^{-1}$, we can select the coefficient a_2, $-1 < a_2 < 1$, so that the equality

$$p(a_1 e_1 + a_2 e_2) = (1 + 2^{-2})^{-1}$$

should hold. We consider the closed subspace of codimension two

$$X_2 = \mathrm{Ker}\, f_{e_1} \cap \mathrm{Ker}\, f_{a_1 e_1 + a_2 e_2}.$$

Starting from the same considerations as in the case X_1, we can select an element $e_3 \in \partial V$ so that $f_{e_1}(e_3) = f_{a_1e_1+a_2e_2}(e_3) = 0$ and $p_3(e_3) \leq 4^{-3}$. In this case we again have

$$\max\{p(a_1e_1 + a_2e_2 + e_3), p(a_1e_1 + a_2e_2 - e_3)\} \geqslant 1$$

and we can select the coefficient a_3, $-1 < a_3 < 1$, so that the equality

$$p(a_1e_1 + a_2e_2 + a_3e_3) = (1 + 2^{-3})^{-1}$$

should hold. Continuing this process indefinitely, we obtain a sequence of elements $\{e_n\}_1^\infty$ and of coefficients $\{a_n\}_1^\infty$. Introducing the notation $s_n = \sum_{k=1}^n a_k e_k$, we can write the properties of the selected sequences in the following manner:

$$|a_n| < 1,\ p(e_n) = 1,\ p_n(e_n) \leqslant 4^{-n};$$
$$f_{s_k}(e_n) = 0\ (1 \leqslant k < n),$$
$$p(s_n) = (1 + 2^{-n})^{-1},\quad n = 1, 2, \ldots$$

From here it follows, in particular, that the series $\sum_{n=1}^\infty a_n e_n$ converges in the topology of the space X and, by virtue of the fact that V is closed, its sum s belongs to the set V: $p(s) \leq 1$. From the definition of the functionals f_{s_n} we obtain the following inequalities:

$$p(s) \geqslant f_{s_n}(s) = f_{s_n}(s_n) = p(s_n) = (1 + 2^{-n})^{-1},\ n = 1, 2, \ldots$$

Consequently, $p(s) = 1$, i.e., $s \in \partial V$. Now we prove that s is an unattainable point of the set V. We select an arbitrary linear functional $f \in X'$ such that $f(s) > 0$. We form a nonincreasing sequence of quantities

$$p_n(f) = \sup\{f(x)\colon x \in X,\ p_n(x) \leqslant 1\}.$$

Let n_0 be the index starting with which $p_n(f) < \infty$. Without loss of generality, we can assume that $p_{n_0}(f) = 1$. We select $m \geq n_0$ in such a manner that the inequalities

$$f(s_m) > \frac{1}{2} f(s) > 2^{-m} \tag{1}$$

should hold. We define the element $y \in \partial V$ in the following manner:

$$y = s_m / p(s_m) = (1 + 2^{-m}) s_m. \tag{2}$$

We compare the values of f(s) and f(y):

$$f(s) = f(s_m) + \sum_{k=m+1}^\infty a_k f(e_k) \leqslant f(s_m) + \sum_{k=m+1}^\infty |a_k| \cdot p_k(f) \cdot p_k(e_k),$$

and, since $|a_k| < 1$, $p_k(f) \leqslant p_{n_0}(f) = 1$, $\sum p_k(e_k) < 4^{-m}$, we have

$$f(s) \leqslant f(s_m) + 4^{-m}.$$

If we now make use of (1) and (2), then we obtain that

$$f(s) < f(s_m) + 2^{-(m+1)} f(s) < f(s_m) + 2^{-m} f(s_m) = f(y).$$

Thus, no linear functional $f \in X'$ attains at the point $s \in \partial V$ its supremum on the set V. Contradiction. Consequently, V has a nonempty interior. Now, finally, we make use of the boundedness of the neighborhood V of zero and by Kolmogorov's theorem we obtain that V defines the initial topology of the space X. The theorem is proved.

In conclusion, we give an example of a complete, barrelled, convex space, to which Theorem 1 cannot be extended.

James [6] has constructed an example of an incomplete normed space E_0, in which each continuous linear functional attains its norm on the unit ball [i.e., the norm of the space E_0 has property (J)]. We denote the norm of the space E_0 by $p_0(\cdot)$. We introduce in E_0 the strongest locally convex topology and we denote the obtained locally convex space by E; it is known [7, 8] that E is complete and barrelled. We obtain: in the complete, barrelled, locally convex space E there exists a continuous norm with the property (J); however, E is not a normed space. Finally, we show that the conjugate space E' is a counterexample to the possibility of the extension of Theorem 2. We select in E' a set V, the polar of the set $U = \{x \in E\colon p_0(x) \leqslant 1\}$. We obtain: in the complete, barrelled, locally convex space X = (E', σ(E', E)), there exists a bounded, closed, absolutely convex set V with property (J*) which, however, does not define the initial topology in X.

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A METHOD OF CONSTRUCTING FAST ALGORITHMS IN THE k-VALUED LOGIC

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Various problems in multivalued logics, in particular problems related to functional completeness, often require solving the question on membership of a given function in a given closed class [1]. The present article describes a general method of constructing fast algorithms for detecting whether functions of multivalued logic belong to classes defined by predicates, and produces examples of applications of this method.

We denote by E_k the set $\{0, 1, \ldots, k-1\}$ and by E^n_k the set of all n-tuples $\tilde{\alpha}^n = (\alpha_1, \ldots, \alpha_n)$ with elements in E_k. The family of all functions $f(x_1, \ldots, x_n)$ mapping E^n_k, for some natural number n, to E_k is denoted by P_k. Suppose that a function $f(\tilde{x}^n)\colon E^n_k \to E_k$ and an l-ary predicate $R(\tilde{x}^l)\colon E^l_k \to E_2$ are given. We say that the function $f(\tilde{x}^n)$ preserves the predicate $R(\tilde{x}^l)$ if for any l n-tuples $\tilde{\alpha}^n_i = (\alpha_{i1}, \ldots, \alpha_{in})$ $(i = 1, \ldots, l)$ holds the implication

$$(\forall j R(\alpha_{1j}, \ldots, \alpha_{lj}) = 1) \Rightarrow R(f(\tilde{\alpha}_1^n), \ldots, f(\tilde{\alpha}_l^n)) = 1. \quad (1)$$

The family of all functions preserving a predicate $R(\tilde{x}^l)$ is a closed class in P_k relative to the superposition operation and is called the preservation class of the predicate $R(\tilde{x}^l)$. We will note this class by κ(R).

Below, we will describe a method of constructing algorithms for detecting the membership of functions $f(\tilde{x}^n) \in P_k$ in the class κ(R); here, the predicate $R(\tilde{x}^l)$ need not be fixed, i.e., it may be given at the input of the algorithm together with the function $f(\tilde{x}^n)$. This variant of the problem arises, for instance, when it becomes necessary to construct algorithms for recognizing membership of functions in classes defined by predicates within some growing set. We will perform an estimation of time complexity of algorithms obtained by the method described. The time complexity of an algorithm means dependence of the number of steps used by

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