CONDITIONS FOR THE CONVEXITY OF THE LIMIT SET OF RIEMANN SUMS OF A VECTOR-VALUED FUNCTION

V. M. Kadets and M. I. Kadets

Let there be given a bounded function f from the segment [0, 1] into a Banach space and suppose that Γ is a partition of this segment by the points a_k and b_k:

$$0 = a_1 < b_1 = a_2 < b_2 = \ldots < b_N = 1.$$

In each interval (a_j, b_j) we select a point x_j. The vector from X that is equal to $\sum_1^N f(x_j)\Delta_j$, where $\Delta_j = b_j - a_j$, is called the Riemann sum $\sigma_f(\Gamma, \{x_j\})$, corresponding to the given function f, the given partition Γ, and the selected set $\{x_j\}$. We will say that a partition Γ is finer than ε if $\Delta_j < \varepsilon$ for arbitrary j. A point $y \in X$ is called a limit power of the Riemann sums of f if for each ε > 0 there exist a partition Γ that is finer than ε and a set of points $\{x_j\}$ such that $\| \sigma_f(\Gamma, \{x_j\}) - y \| \leqslant \varepsilon$. Let us denote the set of all limit points of the Riemann sums of the function f by $\mathcal{J}(f)$. If X is the real axis, then $\mathcal{J}(f)$ is the segment joining the lower and the upper Riemann integrals of f. As proved in [1], if X is a finite-dimensional space or a Hilbert space, then $J(f)$ is a convex set.

The aim of the present article is, in the first place, to extend the result of [1] to B-convex Banach spaces and, secondly, to give an example of a bounded function g from the segment [0, 1] into the space l_1 such that $\mathcal{J}(g)$ is not a convex set.

We will say that a Banach space Y has type p with a constant C if for each set of vectors $y_1, y_2, \ldots, y_n \in Y$ there exists a set of numbers $\xi_1, \xi_2, \ldots, \xi_n = \pm 1$ such that

$$\left\| \sum_{k=1}^n \xi_k y_k \right\|^p \leqslant C \sum_{k=1}^n \| y_k \|^p.$$

A Banach space is said to be B-convex if it has type p > 1. For example, all spaces L_p for $1 < p < \infty$ are B-convex.

<u>THEOREM 1.</u> Let a Banach space Y have type p > 1 with a constant C. Let a function $f\colon [0; 1] \to Y$ be such that $\| f(x) \| \leqslant k < \infty$ for each x in its domain. Then $\mathcal{J}(f)$ is a convex set.

<u>Proof.</u> Let y_1 and y_2 belong to $\mathcal{J}(f)$. We prove that $(y_1 + y_2)/2 \in \mathcal{J}(f)$. Let there be given an N > 1. We select partitions Γ_1 and Γ_2, finer than 2^{-2N}, and point sets $\{x_k^1\}$ and $\{x_k^2\}$, such that $\| \sigma_f(\Gamma_1, \{x_k^1\}) - y_1 \| \leqslant 2^{-N}$ and $\| \sigma_f(\Gamma_2, \{x_k^2\}) - y_2 \| \leqslant 2^{-N}$. Let y_1^j $(1 \leqslant j \leqslant 2^N)$ denote the part of the sum $\sigma_f(\Gamma_1, \{x_k^1\})$, corresponding to the segments, whose right endpoints are less than $j/2^N$, and the left endpoints are greater than $(j-1)/2^N$. Then

$$\left\| \sum_{j=1}^{2^N} y_1^j - \sigma(\Gamma_1, \{x_k^1\}) \right\| \leqslant 2^N \cdot k \cdot 2^{-2N} = \frac{k}{2^N}.$$

We introduce vectors y_2^j in the same manner. The following inequality is fulfilled for them:

$$\left\| \sum_{j=1}^{2^N} y_2^j - \sigma(\Gamma_2, \{x_k^2\}) \right\| \leqslant \frac{k}{2^N}.$$

Let $\{\xi_j\}_1^{2^N}$ be a set of signs such that

$$\left\| \sum_{j=1}^{2^N} \frac{y_1^j - y_2^j}{2} \xi_j \right\|^p \leqslant C \sum_{j=1}^{2^N} \left\| \frac{y_1^j - y_2^j}{2} \right\|^p \leqslant C \sum_{j=1}^{2^N} \left(\frac{k}{2^{N+1}} \right)^p = \frac{C \cdot k^p}{2^p \cdot 2^{(p-1)N}}.$$

Rostov Civil-Engineering Institute. Translated from Matematicheskie Zametki, Vol. 35, No. 2, pp. 161-167, February, 1984. Original article submitted February 8, 1983.

Let us construct a partition Γ_3, finer than 2^{-N}, and select points $\{x_k^3\}$, such that if $\xi_j = +1$, then the partition Γ_3 coincides with Γ_1 and x_k^3 coincides with x_m^1 on the segment $((j-1)/2^N; j/2^N)$ and if $\xi_j = -1$, then Γ_3 and $\{x_k^3\}$ coincide with Γ_2 and $\{x_k^2\}$ respectively on this segment. Then

$$\left\|\sigma_f(\Gamma_3, \{x_k^3\}) - \frac{1}{2}\sum_{j=1}^{2^n}(y_1^j + y_2^j)\right\| \leqslant \frac{k}{2^N} + \left\|\sum_1^{2^N} \xi_j \frac{y_1^j - y_2^j}{2}\right\| \leqslant \frac{k}{2^N} + \left(\frac{c\cdot k^p}{2^p\cdot 2^{(p-1)N}}\right)^{1/p}.$$

Consequently,

$$\left\|\sigma_f(\Gamma_3, \{x_k^3\}) - \frac{y_1 + y_2}{2}\right\| \leqslant \frac{1}{2^N} + 2\frac{k}{2^N} + \left(\frac{c\cdot k^p}{2^p\cdot 2^{(p-1)N}}\right)^{1/p}.$$

The right-hand side of the last inequality converges to zero as $N \to \infty$. Consequently, $(y_1 + y_2)/2 \in \mathcal{J}(f)$. Since $\mathcal{J}(f)$ is closed, it follows that it is a convex set. The theorem is proved.

Now, we begin with the construction of a function with nonconvex set of limit points of the Riemann sums. We fix $\varepsilon = 10^{-6}$; and introduce in our consideration the function $\Psi(t)$, the distance of t from the nearest integral point, which is defined on the whole axis. We select two denumerable families of points on the segment [0, 1]: $T^1 = \bigcup_{n=1}^{\infty} T_n^1$ and $T^2 = \bigcup_{n=1}^{\infty} T_n^2$, where $T_n^j = \{t_{n,k}^j\}$ $(j = 1, 2;\ 1 \leqslant k \leqslant 2^n)$ is the set of roots of the equation

$$\frac{1}{2^n}\Psi(2^n\cdot t) = \frac{\sqrt{j+1}}{4^n}\cdot\varepsilon^2$$

that lie on the segment [0, 1]. Let us observe that the sets T_n^j are pairwise disjoint. Let e(1), e(2), . . . denote the unit vectors of the canonical basis of the space l_1 and let us define the desired function g(t) from [0, 1] into l_1 in the following manner:

$$g(t) = \begin{cases} e(1) \quad \text{for} \quad t \notin T^1 \cup T^2, \\ e(2) + \left[e(2^{n+1} + 2k) - \frac{1}{2^n}\sum_{i=1}^{2^n} e(2^{n+1} + 2i)\right] \\ \qquad\qquad \text{for} \quad t = t_{n,k}^1, \\ e(3) + \left[e(2^{n+1} + 2k - 1) - \frac{1}{2^n}\sum_{i=1}^{2^n} e(2^{n+1} + 2i - 1)\right] \\ \qquad\qquad \text{for} \quad t = t_{n,k}^2. \end{cases}$$

<u>THEOREM 2.</u> $J(g)$ is not a convex set.

<u>Proof.</u> If we take the partition of the segment [0, 1] into 2^n equal segments as Γ and the points $t_{n,k}^1$ as x_k, then

$$\sigma_g(\Gamma, \{x_j\}) = \frac{1}{2^n}\sum_{j=1}^{2^n}\left[e(2) + e(2^{n+1} + 2j) - \frac{1}{2^n}\sum_{i=1}^{2^n} e(2^{n+1} + 2i)\right] = e(2).$$

Consequently, $e(2) \in \mathcal{J}(g)$. Analogously, $e(3) \in \mathcal{J}(g)$. But, as we now show, $[e(2) + e(3)]/2 \notin \mathcal{J}(g)$. Indeed, suppose that for a certain partition Γ and a certain choice of $\{x_j\}$ we have

$$\left\|\sigma_g(\Gamma, \{x_j\}) - \frac{e(2) + e(3)}{2}\right\| \leqslant \varepsilon^2.$$

Let us represent $\sigma_g(\Gamma, \{x_k\}) = \Sigma g(x_k)\Delta_k$ in the form

$$\sum\nolimits_1 g(x_k)\Delta_k + \sum\nolimits_2 g(x_k)\Delta_k + \sum\nolimits_3 g(x_k)\Delta_k,$$

where the first sum is formed from all those terms for which $x_k \in T^1$, and the second sum is formed from all those terms for which $x_k \in T^2$, and the sum Σ_3 is formed from all those terms for which $x_k \notin T^1 \cup T^2$. We get

$$\varepsilon^2 \geqslant \left\|\sigma_g(\Gamma, \{x_k\}) - \frac{e(2) + e(3)}{2}\right\| = \left\|\sum\nolimits_3 g(x_k)\Delta_k\right\| + \left\|\frac{e(2)}{2} - \sum\nolimits_1 g(x_k)\Delta_k\right\| + \left\|\frac{e(3)}{2} - \sum\nolimits_2 g(x_k)\Delta_k\right\|.$$

Consequently, $\Sigma_3\Delta_k \leqslant \varepsilon^2$;

$$\left\|\frac{e(2)}{2}-\sum\nolimits_1 g(x_k)\,\Delta_k\right\|\leqslant\varepsilon^2;\quad \left\|\frac{e(3)}{2}-\sum\nolimits_2 g(x_k)\,\Delta_k\right\|\leqslant\varepsilon^2.$$

Now, we investigate the sum $\sum_1$ separately. Let us represent $\sum_1$ in the form $\sum_1^1 g(x_k)\Delta_k+\sum_1^2 g(x_k)\Delta_k+\ldots+\sum_1^n g(x_k)\Delta_k$, where all the terms for which $x_k\in T_m^1$ occur in $\sum_1^m$. Let the set of m such that $\sum_1^m\Delta_k\neq 0$ be denoted by A. Then

$$\left\|\sum\nolimits_1 g(x_k)\Delta_k-\frac{e(2)}{2}\right\|=\left\|\sum\nolimits_1\Delta_k e(2)-\frac{e(2)}{2}+\sum\nolimits_{n\in A}\sum\nolimits_1^n[g(x_k)-e(2)]\,\Delta_k\right\|=$$

$$=\left|\frac{1}{2}-\sum\nolimits_1\Delta_k\right|+\sum\nolimits_{n\in A}\left\|\sum\nolimits_1^n[g(x_k)-e(2)]\,\Delta_k\right\|.$$

Consequently,

$$\left|\frac{1}{2}-\sum\nolimits_1\Delta_k\right|+\sum\nolimits_{n\in A}\left\|\sum\nolimits_1^n[g(x_k)-e(2)]\,\Delta_k\right\|\leqslant\varepsilon^2. \tag{1}$$

Let B denote the set of those $n\in A$ for which

$$\left\|\sum\nolimits_1^n[g(x_k)-e(2)]\,\Delta_k\right\|>\varepsilon\cdot\sum\nolimits_1^n\Delta_k.$$

It follows from the inequality (1) that $\sum_{n\in B}\sum_1^n\Delta_k<\varepsilon$. Let C denote the set $A\setminus B$. The following inequalities are fulfilled:

$$\sum\nolimits_{n\in C}\sum\nolimits_1^n\Delta_k>\frac{1}{2}-\varepsilon-\varepsilon^2, \tag{2}$$

$$\left\|\sum\nolimits_1^n[g(x_k)-e(2)]\,\Delta_k\right\|\leqslant\varepsilon\sum\nolimits_1^n\Delta_k \quad (\text{for all } n\in C). \tag{3}$$

Let $R_1(n)$ be equal to 2^n minus the number of terms in the sum Σ_1^n. We will define the function $R_1(n)$ only for $n\in C$. The inequality (3) can be rewritten in the following form: For all $n\in C$

$$\sum\nolimits_1^n\left|\Delta_k-\frac{1}{2^n}\sum\nolimits_1^n\Delta_j\right|+R_1(n)\cdot\frac{1}{2^n}\sum\nolimits_1^n\Delta_j\leqslant\varepsilon\sum\nolimits_1^n\Delta_j. \tag{4}$$

It follows from (4) that for all $n\in C$, at least $(1-\varepsilon)\cdot 2^n$ elements of T_n^1 "occur" as x_k in the sum $\sum_1^n g(x_k)\Delta_k$.

Now, let us recall how the set T_n^1 is constituted. This set consists of 2^n points of the segment [0, 1] such that a point of the set T_n^1 is situated at a distance $4^{-n}\varepsilon^2\sqrt{2}$ on the left and on the right of each point of the form $k/2^{n-1}$, i.e., each point of the form $k/2^{n-1}$ is squeezed between two points of the set T_n^1.

Let $n\in C$, and m > n. A segment (a_k, b_k) from $\sum_1^m$ will be said to be squeezed between points from $\sum_1^n$, if there exist two points x_r and x_j in $\sum_1^n$ such that $x_r\leqslant a_k<b_k\leqslant x_j$, and $\|x_r-x_j\|=2\cdot 4^{-n}\cdot\varepsilon^2\cdot\sqrt{2}$. The total length of the segments from $\sum_{m>n}\cdot\sum_1^m$ that are squeezed between the points x_k from $\sum_1^n$ does not exceed $2^n\cdot 4^{-n}\cdot\varepsilon^2\cdot\sqrt{2}=2^{-n}\varepsilon^2\sqrt{2}$. Therefore, the total length of all the "squeezed" segments is less than $\sum_{n\in C}2^{-n}\varepsilon^2\sqrt{2}<2\varepsilon^2$. Let us combine all the nonsqueezed segments from $\sum_1^n$ for all n into a sum and denote it by $\sum_{1,n}$. Since the total length of the segments that belong to $\sum_{n\in C}\sum_1^n$, but do not belong to $\sum_{n\in C}\sum_{1,n}$, does not exceed $2\varepsilon^2$, it follows from (1) that

$$\sum\nolimits_{n\in C}\left\|\sum\nolimits_{1,n}[g(x_k)-e(2)]\,\Delta_k\right\|\leqslant 10\varepsilon^2, \tag{5}$$

and it follows from (2) that

$$\sum_{n\in C}\sum_{1,\,n}\Delta_k > \frac{1}{2} - \varepsilon - \varepsilon^2 - 2\varepsilon^2. \tag{6}$$

Let C_1 denote the set of all $n \in C$ such that

$$\left\|\sum_{1,\,n}[g(x_k) - e(2)]\,\Delta_k\right\| \leqslant \varepsilon \sum_{1,\,n}\Delta_k. \tag{7}$$

In the same way as we have obtained (2) from (1), we get the following inequality from (5) and (6):

$$\sum_{n\in C_1}\sum_{1,\,n}\Delta_j > \frac{1}{2} - 15\cdot\varepsilon - 3\cdot\varepsilon^2.$$

Let us define $R_2(n)$ as 2^n minus the number of terms in the sum $\Sigma_{1,n}$. Analyzing the inequality (7) in the same manner as we have analyzed the inequality (3) earlier, we get

$$R_2(n) \leqslant \varepsilon\cdot 2^n \tag{8}$$

for all $n \in C_1$. Let N denote $\max\{n\colon n \in C_1\}$. We prove that the sum $\sum_1 g(x_k)\,\Delta_k$ is practically equal to the sum $\sum_{1,N} g(x_k)\Delta_k$.

Let $n < N$ and $n \in C_1$. At least $2^n\,(1 - 2\varepsilon)$ points of T^1_N are squeezed between the points from $\Sigma_{1,n}$. Since all the "squeezed" segments have been deleted from $\sum_{1,N}$ it follows that $R_2(N) \geqslant 2^n\,(1 - 2\varepsilon)$. Hence from (8) we get

$$2^N\frac{\varepsilon}{1-2\varepsilon} \geqslant 2^n. \tag{9}$$

We decompose $\sum_{n\in C_1\setminus\{N\}}\sum_{1,\,n}\Delta_j$ into two sums: $\sum^1$ and $\sum^2$. In $\sum^1$ we combine all the "small" segments (such that $\Delta_j < \frac{1}{2^{N-2}}$), and in $\sum^2$ we combine the remaining ones (i.e., "large" segments). From (9) we get

$$\sum{}^1\Delta_k < \sum_{n\in C_1\setminus\{N\}}\sum_{1,\,n}\frac{1}{2^{N-2}} < 2^N\frac{\varepsilon}{1-2\varepsilon}\frac{2}{2^{N-2}} < 10\varepsilon. \tag{10}$$

Each segment of length Δ_j from $\sum^2$ covers at least $\Delta_j\cdot 2^{N-1}$ points of T^1_N. Therefore, $R_2(n) \geqslant \frac{1}{2}\,2^N\sum^2\Delta_j$. It follows from (8) that $\sum^2\Delta_j < 2\varepsilon$. Hence it follows from (10) that $\sum^1\Delta_j + \sum^2\Delta_j < 12\varepsilon$. Consequently, $\sum_{1,\,N}\cdot\Delta_j > \frac{1}{2} - 30\varepsilon$, i.e., $\sum_1$ is practically equal to the sum $\sum_{1,\,N}$.

In the same manner, we can introduce the sums $\Sigma_{2,\,n}$ and in exactly the same manner we can show that Σ_2 is practically equal to the sum $\Sigma_{2,\,M}$. Let $N > M$. In the same manner as the above arguments, we get

$$2^M \leqslant \frac{\varepsilon}{1-2\varepsilon}\,2^N \text{ and } \sum_{2,\,M}\Delta_k < 12\varepsilon.$$

This is a contradiction. Consequently, $[e(2) + e(3)]/2 \notin \mathcal{J}(g)$. The theorem is proved.

LITERATURE CITED

1. I. Halperin and N. Miller, "An inequality of Steinitz and the limits of Riemann sums," Trans. R. Soc. Can., 48, 27-29 (1954).