SUBSPACES OF l_1 WITH STRICTLY CONVEX NORM

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In this paper we shall consider the real Banach space c_0 and its adjoint l_1. We denote the canonical basis in l_1 by $\{e_n\}_1^\infty$. Consequently, every element $x \in l_1$ has a unique expansion $x = \sum a_n e_n$, $\|x\| = \sum |a_n|$. The following remark is due to Lindenstrauss [1]: if we take two elements $x = \sum a_n e_n$ and $y = \sum b_n e_n$ in l_1 such that the quotients a_n/b_n are everywhere dense on the real line, then the two-dimensional subspace spanned by x and y is strictly convex. The goal of this paper is to show that l_1 has an infinite-dimensional strictly convex subspace which is closed in the weak-star topology. This implies, in particular, that c_0 has an infinite-dimensional smooth quotient space. Moreover, we show that every strictly convex subspace of l_1 has a stronger property of convexity — it is an MLUR space.

We introduce some definitions. A Banach space X is called strictly convex if its unit sphere S(X) contains no line segments. X is called midpoint locally uniformly convex (notation: $X \in \mathrm{MLUR}$) if the conditions $\lim \|x \pm z_n\| = \|x\|$, $x \in S(X)$, $z_n \in X$, imply that $\lim \|z_n\| = 0$. The space X is called locally uniformly convex (notation: $X \in \mathrm{LUR}$) if the conditions $\lim \|x + x_n\| = 2$, $x, x_n \in S(X)$, imply that $\lim \|x_n - x\| = 0$. The property MLUR has been introduced by Anderson [2]. Concerning its relation to other properties of convexity, see [3-5]. The notions introduced above can be "localized." A point x on the unit sphere is called an extremal point ($x \in \operatorname{ext} S(X)$) if it is not the midpoint of any line segment lying entirely on the sphere. The point x is called an MLUR point if the conditions $\lim \|x \pm z_n\| = 1$ imply that $\lim \|z_n\| = 0$. Finally, a Banach space is called smooth if there exists only one supporting hyperplane at every point of the unit sphere. Concerning other equivalent definitions of the notions introduced above and concerning relations of duality, etc., see [6].

LEMMA. A subspace E of l_1 is strictly convex if and only if for any pair of linearly independent elements $x, y \in E$, there exists an index m such that $a_m \cdot b_m < 0$, ($x = \sum a_m e_m$, $y = \sum b_m e_m$).

Proof. We must compare the numbers

$$\|x + y\| = \sum |a_n + b_n|, \quad \|x\| = \sum |a_n|, \quad \|y\| = \sum |b_n|.$$

It is clear that the inequality $\|x + y\| < \|x\| + \|y\|$ implies that for at least one m we have $a_m \cdot b_m < 0$. Conversely, if $a_n \cdot b_n > 0$ then $\|x + y\| = \|x\| + \|y\|$ for any n.

THEOREM 1. There exists an infinite-dimensional strictly convex subspace of l_1 which is closed in the weak-star topology.

Proof. We index all nonzero rational numbers in such a way that for the sequence $\{r_n\}$ thus obtained, the series $\sum 2^{-n} |r_n| = A$ converges. We partition the set $\mathbf{N}$ of natural numbers into an infinite number of disjoint infinite sets N_{jk} (j, k = 1, 2, . . .) and define a surjection $F\colon \mathbf{N} \to \mathbf{N}$ so that the restriction of F to each N_{jk} is a bijection of N_{jk} onto $\mathbf{N}$. Put

$$u_{jk} = \sum_{m \in N_{jk}} 2^{-F(m)} e_m; \quad v_{jk} = \sum_{m \in N_{jk}} 2^{-F(m)} r_{F(m)} e_m. \tag{1}$$

It is easy to see that for any λ_{jk},

$$\left\|\sum_{j,k} \lambda_{jk} u_{jk}\right\| = \sum_{j,k} |\lambda_{jk}|, \quad \left\|\sum_{j,k} \lambda_{jk} v_{jk}\right\| = A \sum_{j,k} |\lambda_{jk}|.$$

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Now let

$$g_j = \sum_{k=1}^{\infty} 2^{-k} u_{jk} + \frac{\theta}{j-1} \sum_{i=1}^{j-1} v_{j-i,\,i} \quad \left(\theta = \frac{1}{3A}, \quad j = 1, 2, \ldots\right). \tag{2}$$

It is easy to verify that

$$\operatorname{supp} g_j \cap \operatorname{supp} g_n = N_{j,\,n-j} \qquad (j < n), \tag{3}$$

where supp x denotes the set of those indices for which the coefficients in the canonical expansion of x are different from zero. It is also easy to see that

$$\sum_n a_n g_n = \sum_n \sum_{k=1}^{\infty} \left(2^{-k} a_n u_{nk} + \frac{\theta}{n+k-1} a_{n+k} v_{nk}\right).$$

We estimate the norm of $\sum a_n g_n$ from below:

$$\left\| \sum a_n g_n \right\| = \sum_n \sum_{k=1}^{\infty} \left\| 2^{-k} a_n u_{nk} + \frac{\theta}{n+k-1} a_{n+k} v_{nk} \right\| \geqslant \sum_n \left\| 2^{-1} a_n u_{n1} + \frac{\theta}{n} a_{n+1} v_{n1} \right\| \geqslant$$

$$\geqslant \sum_n \left(\| 2^{-1} a_n u_{n1} \| - \left\| \frac{\theta}{n} a_{n+1} v_{n1} \right\| \right) \geqslant (2^{-1} - \theta A) \sum_n | a_n | = \frac{1}{6} \sum | a_n |.$$

Now we obtain an upper estimate for $\| \sum a_n g_n \|$:

$$\left\| \sum a_n g_n \right\| \leqslant \sum | a_n | \, \| g_n \| \leqslant (1 + \theta A) \sum | a_n | = (4/3) \sum | a_n |.$$

Therefore, the sequence $\{g_n\}_1^{\infty}$ is equivalent to the natural basis of l_1. Let E be the subspace spanned by the system $\{g_n\}_1^{\infty}$. Take two linearly independent vectors $x = \sum a_i g_i$, $y = \sum b_i g_i$ from E. By their linear independence, there are two indices p < q for which

$$a_p b_q - a_q b_p \neq 0. \tag{4}$$

From the expansions of x and y in the canonical basis, we choose those terms whose indices belong to $N_{p,\,q-p}$. According to (2) and (3), these are the elements

$$a_p 2^{-(q-p)} u_{p,\,q-p} + a_q \frac{\theta}{q-1} v_{p,\,q-p},$$

$$b_p 2^{-(q-p)} u_{p,\,q-p} + b_q \frac{\theta}{q-1} v_{p,\,q-p},$$

or finally, according to (1),

$$\sum_{n \in N_{p,\,q-p}} \left(a_p 2^{-(q-p)} + a_q \frac{\theta}{q-1} r_{F(n)} \right) 2^{-F(n)} e_n,$$

$$\sum_{n \in N_{p,\,q-p}} \left(b_p 2^{-(q-p)} + b_q \frac{\theta}{q-1} r_{F(n)} \right) 2^{-F(n)} e_n.$$

By condition (4) and the density of $\{r_{F(n)},\ n \in N_{p,\,q-p}\}$ on the entire real line, there exists an index $m \in N_{p,\,q-p}$ such that

$$\left(a_p 2^{-(q-p)} + a_q \frac{\theta}{q-1} r_{F(m)} \right) \cdot \left(b_p 2^{-(q-p)} + b_q \frac{\theta}{q-1} r_{F(m)} \right) < 0.$$

Consequently, E is strictly convex by Lemma 1. Now we show that E is weak-star closed in l_1. The construction of $\{g_n\}$ implies immediately that g_n weak-star converges to zero. Consider the weak-star closed convex hull K of $G = \{\pm g_n\}_1^{\infty}$. Since the extremal points of K must be among $\{\pm g_n\}$ and the weak-star limit points of G (by the Milman "converse" of the Krein–Milman theorem (see [7, p. 16])), we must have K = G. Consequently, by Choquet's theorem,

$$K = \{\textstyle\sum_1^{\infty} c_n g_n \colon \sum | c_n | \leqslant 1\}.$$

Since $\{g_n\}$ is equivalent to the canonical basis of l_1, the set K contains a ball of E. According to a corollary to the Krein–Milman theorem [8, p. 465, Corollary 8], E is weak-star closed.

COROLLARY. The space c_0 contains a subspace H such that the quotient space c_0/H is smooth.

Proof. It is sufficient to take $H = E_{\perp}$, the annihilator of E in c_0. Then $(c_0/H)^* = E$ and the smoothness of c_0/H follows from the strict convexity of E.

THEOREM 2. Let X be a subspace of l_1. Every extremal point of the unit sphere of X is an MLUR point of it. In particular, if X is a strictly convex subspace, then $X \in$ MLUR.

Proof. Let some point $x \in \text{ext}\, S(X)$ be not an MLUR point. This means that in X there exist sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\inf_n \| u_n - v_n \| > 0, \quad \lim \| u_n \| = \lim \| v_n \| = 1;$$
$$x = (u_n + v_n)/2 \quad (n = 1, 2, \ldots);$$
$$u_n = \sum p_i^{(n)} e_i, \quad v_n = \sum q_i^{(n)} e_i, \quad x = \sum a_i e_i.$$

Since $x \in \text{ext}\, S(X)$ it is easy to see that neither $\{u_n\}$ nor $\{v_n\}$ contains norm convergent subsequences. Using the well-known criterion of compactness in a space with basis, from this we obtain that there exist a subsequence $\{n_k\}_{k=1}^{\infty}$ of indices and a number $\delta > 0$ such that for all k = 1, 2, . . .

$$\sum_{i=k+1}^{\infty} | p_i^{(n_k)} | > \delta, \quad \sum_{i=k+1}^{\infty} | q_i^{(n_k)} | > \delta.$$

Choose the index k_0 so large that for all $k > k_0$,

$$\| u_{n_k} \| \leqslant 1 + \delta/2, \quad \| v_{n_k} \| \leqslant 1 + \delta/2.$$

This implies

$$\sum_{i=1}^{k} | p_i^{(n_k)} | \leqslant 1 - \delta/2, \quad \sum_{i=1}^{k} | q_i^{(n_k)} | \leqslant 1 - \delta/2 \quad (k = k_0 + 1, \ldots). \tag{5}$$

On the other hand, for all k = 1, 2, . . .,

$$(1/2) \sum_{i=1}^{k} | p_i^{(n_k)} + q_i^{(n_k)} | = \sum_{i=1}^{k} | a_i |,$$

and, consequently, $\lim_k (1/2) \sum_{i=1}^{k} | p_i^{(n_k)} + q_i^{(n_k)} | = 1$, since $\|x\| = 1$. The last assertion contradicts (5). The theorem is proved.

We also note that no infinite-dimensional subspace X of l_1 is locally uniformly convex. Indeed, X contains an "almost disjoint" normalized sequence $\{x_n\}_1^{\infty}$ such that

$$\lim_n \| x_1 \pm x_n \| = \lim_n (\| x_1 \| + \| x_n \|) = 2,$$

which contradicts the definition of an LUR space.

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