Remark. According to 1), we obtain a Lagrangian family $\{g\mathcal{F}g^{-1}\}$ on the orbit Ω which enables us to apply the scheme of Sec. 2 to M = Ω and X = G/P.

In contrast to the case of ordinary polarizations, the following theorem holds:

THEOREM 4. Given any $f \in \mathfrak{g}^*$, one can find a subalgebra $\mathfrak{p}$, such that $(f, \mathfrak{p})$ is a polarizing pair.

Now suppose that $\mathfrak{g}$ is semisimple, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra, $n \in \mathfrak{g}$ is a nilpotent element, and $\mu: T^*(G/B) \to \mathfrak{g}$. From Theorem 3 we derive:

THEOREM 5. a) $G \cdot n \cap \mathfrak{n}$ is a Lagrangian manifold and the dimension of the components of the fiber $\mu^{-1}(n)$ is $\dim \mathfrak{n} - {}^1/_2 \cdot \dim G \cdot n$ (see [4]).

b) Every component of $\mu^{-1}(G \cdot n \cap \mathfrak{n})$ is the closure of the conormal bundle of some Schubert cell.

LITERATURE CITED

1. V. Guillemin and S. Sternberg, Am. J. Math., 101, No. 4, 915-955 (1979).
2. V. Guillemin and S. Sterbnerg, Invent. Mat., 67, No. 3, 515-538 (1982).
3. V. Ginzburg, "Symplectic geometry and representations," Preprint, Moscow State Univ. (1981).
4. N. Spaltenstein, Topology, 16, No. 2, 203-204 (1977).

TWO THEOREMS ON THE MASSIVENESS OF BOUNDARIES IN REFLEXIVE BANACH SPACES

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Let X be a reflexive Banach space. We shall say that a subset B of the unit sphere S(X) of X is a boundary (for the dual space X*) whenever

$$\forall f \in X^* \ \exists x \in B: \ f(x) = \|f\|.$$

Obviously, one of the boundaries is the set ext U(X) of extremal points of the unit ball U(X) of X. Generally speaking, however, a boundary does not necessarily contain all the extremal points [although it contains all the exposed points, exp U(X)]. The theorem of Lindenstrauss and Phelps [1] asserting that the set of extremal points of the unit ball in a reflexive Banach space is not countable has stimulated a series of papers devoted to the investigation of the massiveness (in some sense or another) of this set ([2, 3]). One can show that a boundary also enjoys the same massiveness properties.

The goal of this note is to discuss two new natural massiveness properties and prove that they are enjoyed by any boundary in a reflexive Banach space.

Proposition 1. Let X be a Banach space and $B \subset S(X)$. Then the following are equivalent statements:

(I) For any Banach space Y and any bounded linear operator T:Y → X such that $T(Y) \supset B$, one has T(Y) = X.

(II) For every representation of B as the union of an increasing series of sets, B = $\bigcup_1^\infty B_i, (B_i \uparrow)$, there is an index j such that

$$\inf_{f \in S(X^*)} \sup_{x \in B_j} |f(x)| > 0,$$

(i.e., B_j is a norming set for X*).

Proof. (I) (II). We proceed by reductio ad absurdum. Suppose that there is a representation $B = \bigcup B_i, B_i \uparrow$, such that none of the B_i is norming. Set $A_1 = B_1$, $A_i = B_i \setminus B_{i-1}$

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$(i = 2, 3, \ldots)$, $V = \operatorname{cl} \operatorname{conv} \{i^{-1}A_i\}_{i=1}^{\infty}$. Let L be the linear manifold spanned by V, i.e., $L = \bigcup_{n=1}^{\infty} n \cdot V$. Let us show that $L \neq X$. Since V is absolutely convex and closed, it suffices to verify that V contains no ball centered at zero. Let $\varepsilon > 0$ and take j such that $j^{-1} < \varepsilon$. Since B_j is not a norming set, there exists a functional $f \in S(X^*)$ such that $\sup_{x \in B_j} |f(x)| < \varepsilon$. Using the definition of V, we get $\sup_{x \in V} |f(x)| < \varepsilon$, which implies that V does not contain the ball $\varepsilon \cdot U(X)$.

Let Y denote the linear space L with unit ball V. From the fact that V is closed it follows that Y is a Banach space. Let T denote the natural embedding of Y (i.e., L) into X. It is readily seen that T is a bounded linear injection $Y \to X$ such that $T(Y) \supset B$, but $T(Y) = L \neq X$. This contradicts (I) and proves the implication (I) $\Rightarrow$ (II).

(II) $\Rightarrow$ (I). Let T be a bounded linear operator from the Banach space Y into X, such that . Set $B_i = T(i \cdot U(Y)) \cap B$, $i = 1, 2, \ldots$. Obviously, $\{B_i\}$ is an increasing sequence with $B = \bigcup B_i$. But then there is j such that

$$\inf_{f \in S(X^*)} \sup_{x \in B_j} |f(x)| = \delta > 0,$$

which implies immediately that $\operatorname{cl} T(U(Y)) \supset \delta/j\, U(X)$. By Lemma 2 ([4], p. 57), the last inclusion means precisely that T is surjective. The proposition is proved.

THEOREM 1. Suppose X is a reflexive Banach space and B is a boundary. Then for any Banach space Y and any bounded linear operator $T: Y \to X$ such that $T(Y) \supset B$, one has $T(Y) = X$.

Proof. By Proposition 1, it suffices to verify that every boundary in a reflexive Banach space has property (II). To do this one can proceed as in the proof of Theorem 1 from [3]. The theorem is proved.

Remark. It is not known whether one can replace, in Theorem 1, B by the set of exposed points $\exp U(X)$.

The proof of the next result is straightforward.

Proposition 2. Let $\{x_i, f_i\}_1^{\infty}$ be a biorthogonal system in the Banach space X. Set

$$L = \{x = \sum_1^{\infty} a_i x_i : \sum_1^{\infty} a_i x_i \text{ converges in norm}\}$$

and introduce a new norm on the linear manifold L by

$$|||x||| = \sup_n \left\| \sum_1^n a_i x_i \right\|, \quad x = \sum a_i x_i \in L.$$

Then $(L, |||\cdot|||)$ is a complete linear normed space with basis $\{x_i\}$, and

$$\|x\| \leqslant |||x|||, \quad x \in L. \tag{1}$$

THEOREM 2. Suppose X is a reflexive Banach space, B is an arbitrary boundary, and the minimal system $\{x_i\}$ has the property that every element of B can be expanded into a norm-convergent series with respect to $\{x_i\}$. Then $\{x_i\}$ is a basis of the whole X.

Proof. Define the linear manifold L and the norm $|||\cdot|||$ as in Proposition 2. Let $Y = (L, |||\cdot|||)$ and let T be the natural embedding $Y \to X$. By (1), T is a bounded linear operator. Furthermore, $T(Y) \supset B$, by the assumptions of our theorem. From Theorem 1 it now follows that $T(Y) = X$, i.e., $\{x_i\}$ is a basis of X, as claimed. The theorem is proved.

Remark. Let B be an arbitrary boundary in the reflexive Banach space X. It can be shown that there is no minimal system in X with respect to which every element of B can be expanded into an absolutely convergent series.

LITERATURE CITED

1. J. Lindenstraus and R. Phelps, Isr. J. Math., 6, 39-48 (1968).
2. M. I. Kadets and V. P. Fonf, Mat. Zametki, 20, No. 3, 315-319 (1976).
3. V. P. Fonf, Ukr. Mat. Zh., 30, No. 5, 692-695 (1978).
4. N. Bourbaki, Topological Vector Spaces, Addison-Wesley.