SOME PROPERTIES OF THE SET OF EXTREME POINTS OF THE UNIT BALL OF A BANACH SPACE

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In [1] it was proved that the unit ball of a reflexive Banach space has an uncountable set of extreme points. In this note it is shown that this set is also massive in some topological sense.

In [1] the following theorems are proved.

THEOREM A. If E is an infinite-dimensional reflexive Banach space, then the set of extreme points of its unit ball is uncountable.

THEOREM B. Let E be a Banach space and let the unit ball of the space E* have a countable set of extreme points. Then

1) E* is separable, and

2) E does not contain any finite-dimensional reflexive subspaces.

In these theorems the massiveness of the set of extreme points is characterized by its cardinality.

The object of this note is to obtain a generalization of Theorems A and B in which the massiveness of the set of extreme points is characterized in topological terms.

THEOREM 1. Let E be a Banach space and K be a ω*-closed, convex, and bounded subspace of E*. If the set of extreme points of K is "strongly" (i.e., in the norm topology of E*) separable, then K is the closure in the norm topology of the convex hull of its extreme points.

Proof. Let $\{f_j\}_1^\infty$ be a strongly dense subset of ext K and $\varepsilon_n \downarrow 0$. For each natural number n we denote by $\{V_j^n\}_{j=1}^\infty$ the sequence of closed balls of radius ε_n with centers at the points f_j. It is clear that

$$\operatorname{ext} K \subset \bigcup_{j=1}^{\infty} V_j^n = H^n \quad (n = 1, 2, \ldots).$$

Since each ball is a ω*-closed set, H^n is a set of type F_σ. We set

$$A_m^n = V_m^n \setminus \bigcup_{i=1}^{m-1} V_i^n \quad (m, n = 1, 2, \ldots).$$

For each n = 1, 2, . . ., it is obvious that we have

$$A_i^n \cap A_j^n = \phi \quad (i \neq j).$$

Without loss of generality, we may assume that all A_m^n are nonempty. We must prove that any element $f_0 \in K$ can be approximated with any degree of accuracy by convex combinations of extreme points. Since K is a weak* convex compact set, by the Choquet–Bishop–de Leeuw theorem (see [2], the note on p. 33) there exists a probability measure μ which represents f_0 and which for any n is concentrated on $H^n = \bigcup_{m=1}^\infty A_m^n$:

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$$f_0(x)=\int_{H^n} f(x)\,\mu(df) \qquad (x\in E).$$

We take an arbitrary $\delta > 0$ and take n so that $\varepsilon_n < \delta/2$. With the help of the measure μ we form the series $\sum_{m=1}^{\infty}\mu(A_m^n)f_m$ and with it we approximate f_0:

$$\left\| f_0-\sum_{m=1}^{\infty}\mu(A_m^n)\,f_m\right\| = \sup_{\|x\|=1}\left|f_0(x)-\sum_{m=1}^{\infty}\mu(A_m^n)f_m(x)\right| = \sup_{\|x\|=1}\left|\int_{H^n} f(x)\,\mu(df)-\sum_{m=1}^{\infty}\int_{A_m^n} f_m(x)\,\mu(df)\right| =$$

$$= \sup_{\|x\|=1}\left|\sum_{m=1}^{\infty}\int_{A_m^n}(f-f_m)(x)\,\mu(df)\right| \leqslant \sup_{\|x\|=1}\sum_{m=1}^{\infty}\sup_{f\in A_m^n}|(f-f_m)(x)|\,\mu(A_m^n).$$

Since

$$A_m^n \subset V_m^n \quad\text{and}\quad f_m \in V_m^n,$$

each term of the last series satisfies

$$|(f-f_m)(x)|\,\mu(A_m^n)\leqslant \|f-f_m\|\,\mu(A_m^n)\leqslant \delta\mu(A_m^n).$$

Consequently,

$$\left\| f_0-\sum_{m=1}^{\infty}\mu(A_m^n)\,f_m\right\| \leqslant \sum_{m=1}^{\infty}\delta\mu(A_m^n)=\delta.$$

Because δ was arbitrary, this last inequality proves the theorem.

COROLLARY. Let E be a Banach space and let the set of extreme points of the unit ball of the space E* be strongly separable. Then E* is a separable Banach space.

THEOREM 2. If E is an infinite-dimensional reflexive Banach space, then the set of extreme points of its unit ball cannot be covered by a countable collection of sets which are compact in the norm topology.

Proof. We follow the method developed in [1]. We suppose to the contrary that

$$\operatorname{ext} U \subset \bigcup_{i=1}^{\infty} K_i,$$

where each K_i is compact in the norm topology. Without loss of generality, we may assume that

$$K_i \subset U = \{x\in E\colon \|x\|\leqslant 1\} \qquad (i=1,2,\ldots).$$

We note that it follows from the corollary to Theorem 1 that E is a separable Banach space and, consequently, U* (the unit ball of E*) is a metrizable compact set in the w*-topology of E* (which in this case obviously coincides with the w-topology). We set $F_n=\{f\in U^*\colon \exists x\in K_n,\ f(x)=\|f\|\}$ $(n=1,2,\ldots)$. We will prove that each F_n is closed in the w*-topology of E*. Let

$$f^{(k)}\xrightarrow{w^*} f,\quad f^{(k)}\in F_n$$

(n fixed, k = 1, 2, . . .). We will show that $f\in F_n$. For each k there exists a point $x^{(k)}\in K_n$ such that

$$f^{(k)}(x^{(k)})=\|f^{(k)}\|.$$

Due to the compactness of K_n there exists a subsequence $\{x^{(k_l)}\}_{l=1}^{\infty}$ of the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ and a point $x\in K_n$ such that $\|x^{(k_l)}-x\|\to 0$ as $l\to\infty$. We have

$$|f^{(k_l)}(x^{(k_l)})-f(x)|\leqslant |f^{(k_l)}(x^{(k_l)})-f^{(k_l)}(x)|+|f^{(k_l)}(x)-f(x)|\leqslant \|x^{(k_l)}-x\|+|f^{(k_l)}(x)-f(x)|\to 0$$

as $l\to\infty$.

Since

$$\|f\|\leqslant \varliminf_{k}\|f^{(k)}\|\leqslant \varliminf_{l}\|f^{(k_l)}\| = \lim_{l} f^{(k_l)}(x^{(k_l)})=\lim_{l} f^{(k_l)}(x^{(k_l)})=f(x),$$

we have $f(x)=\|f\|$. Consequently, $f\in F_n$, which also proves that F_n is w*-closed. From the reflexiveness of E and the Krein–Milman theorem it follows that

$$U^*=\bigcup_{n=1}^{\infty}F_n.$$

On the basis of Baire's theorem on categories, we can assert that one of the F_n's (say F_1) has a nonempty weak* interior relative to U*. Let F_0 be a weakly* interior point of F_1. Without loss of generality, we may assume that $\|f_0\| = 1 - \delta$, $\delta > 0$. Thus, there exists a finite set $\{x_i\}_{i=1}^m \subset E$ such that if there is a place $f \in U^*$ and

$$\max_{1 \leqslant i \leqslant m} |(f - f_0)(x_i)| < 1,$$

then $f \in F_1$. Let $\{y_j\}_{j=1}^p$ be a finite $\delta/2$-net for K_1; $N = \{f \in E^* : f(x_i) = f_0(x_i),\ i = 1, \ldots, m;\ f(y_j) = f_0(y_j),\ j = 1, \ldots, p\}$. Since E is an infinite-dimensional Banach space, the flat N of finite comeasure contains a straight line passing through f_0 which intersects S* at some point g_0 $(S^* = \{f \in E^* : \|f\| = 1\})$. Thus, $g_0 \in F_1$ and $\|g_0\| = 1$. Consequently, there exists a point $x_0 \in K_1$ such that

$$g_0(x_0) = \|g_0\| = 1.$$

We choose $y_{j_0} \in \{y_j\}_{j=1}^p$ so that $\|x_0 - y_{j_0}\| < \delta/2$. We have

$$|g_0(x_0) - g_0(y_{j_0})| \leqslant \|x_0 - y_{j_0}\| < \delta/2,$$

hence

$$g_0(y_{j_0}) > 1 - \delta/2. \tag{1}$$

But on the other hand, since $g_0 \in N$, we have

$$g_0(y_{j_0}) = f_0(y_{j_0}) \leqslant \|f_0\| = 1 - \delta. \tag{2}$$

Inequalities (1) and (2) are incompatible. This contradiction proves the theorem.

THEOREM 3. Suppose that E is a Banach space and that the set of extreme points of the unit ball of E* can be covered by a countable union of strong compact sets. Then E does not contain any infinite-dimensional reflexive subspaces.

Proof. Suppose that F is an infinite-dimensional subspace of E. Let T be an operator of the isometric enclosure of F in E. Then T* is an epimorphism of E* onto F*, and it is not hard to prove that

$$T^*(U_{E^*}) = U_{F^*},\ T^*(\operatorname{ext} U_{E^*}) \supset \operatorname{ext} U_{F^*}.$$

Let

$$\operatorname{ext} U_{E^*} \subset \bigcup_{n=1}^{\infty} K_n,$$

where each K_n is a compact subset of S_E* in the norm topology of E*. Then

$$\operatorname{ext} U_{F^*} \subset \bigcup_{n=1}^{\infty} T^*(K_n). \tag{3}$$

Since each $T^*(K_n)$ is a strong compact set in F*, (3) contradicts Theorem 2.

COROLLARY. With the hypothesis of Theorem 3, E** is not separable.

The proof follows immediately from a comparison of Theorem 3 and the subsequent result of Rosenthal and Johnson (see [3]). If E** is separable, then E contains an infinite-dimensional reflexive subspace.

LITERATURE CITED

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