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ON ABSOLUTE, PERFECT, AND UNCONDITIONAL CONVERGENCES OF DOUBLE SERIES IN BANACH SPACES

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We prove that, in the case of double series, perfect and unconditional convergences coincide, while absolute and perfect convergences do not coincide even for numerical series.

The aim of the present paper is to extend some concepts and, where possible, results concerning rearrangements of (ordinary) series in Banach spaces to the case of double series. For definiteness, we consider real spaces.

Recall that a series $\sum x_n$ in a Banach space is called absolutely convergent if the numerical series $\sum \|x_n\|$ converges. A series is called perfectly convergent if, for any collection of coefficients $\alpha_n = \pm 1$ (which is denoted as $\alpha \in D$), the series $\sum \alpha_n x_n$ converges. A series is called unconditionally convergent if, for any rearrangement of the natural series $\pi : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum x_{\pi(n)}$ converges perfectly. The converse statement is true if and only if the space is finite-dimensional. For a detailed presentation of the theory of rearrangements of series in Banach spaces, see [1].

Let us pass to the investigation of double series. First of all, we give the following definition of the convergence of double series, which somewhat strengthens the conventional Pringsheim definition of convergence:

Definition 1. *A double series $\sum x_{ij}$ in a Banach space X converges to a sum S if*

(a) the sequence of "rectangular" partial sums converges:

$$S = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}; \tag{1}$$

(b) every "row series" $\sum_j x_{ij}$ converges;

(c) every "column series" $\sum_i x_{ij}$ converges.

We can now introduce the notions of absolute, perfect, and unconditional convergences consistent with the structure of double series as follows:

Definition 2. *A series $\sum x_{ij}$ is called absolutely convergent if the numerical series $\sum \|x_{ij}\|$ converges.*

Definition 3. *A series is called perfectly convergent if, for any collection of coefficients $\alpha, \beta \in D$, the series $\sum \alpha_i \beta_j x_{ij}$ converges.*

Definition 4. *A series is called unconditionally convergent if, for any rearrangement of the natural series π and σ, the series $\sum x_{\pi(i),\sigma(j)}$ converges.*

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Let us show that, in the case of perfect convergence, conditions (b) and (c) in Definition 1 follow from condition (a).

Proposition 1. *Let a double series* $\sum x_{ij}$ *be such that, for any set of coefficients* $\alpha, \beta \in D$, *the sequence of rectangular partial sums is convergent:*

$$S(\alpha, \beta) = \lim S_{m,n}(\alpha, \beta) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}. \tag{1a}$$

Then every column series and every row series converge perfectly.

Proof. Let α and β be arbitrary collections of coefficients. We form a new collection $\alpha' = \{\alpha_1, -\alpha_2, -\alpha_3, \ldots\}$. Then the half sum of the series $\sum \alpha_i \beta_j x_{ij}$ and $\sum \alpha_i' \beta_j x_{ij}$ is a convergent ordinary series $\sum_j x_{1j} \beta_j$. Since $\beta \in D$ is arbitrary, the row series $\sum_j x_{1j} \beta_j$ converges perfectly. The perfect convergence of the other row series and all column series is proved similarly.

Thus, in the investigation of perfect convergence, we can restrict ourselves to the Pringsheim convergence. As shown below, Definition 1 turns out to be more natural for unconditionally convergent series.

Let us formulate the principal results of the present paper.

Theorem 1. *For a double series* $\sum x_{ij}$ *in a Banach space* X, *the following statements are equivalent:*

(A) the series converges perfectly;

(B) the vector-valued matrix $A = (x_{ij})$ *generates a linear compact operator* A *that maps the Banach space* c_0 *into the Banach space* $\mathrm{Perf}(X)$ *of all perfectly convergent ordinary series and acts in the following way:*

$$Ae_j = \bar{x}_j = \{x_{ij}\}_{i=1}^{\infty}, \quad j \in \mathbb{N},$$

where (e_j) *is a canonical basis of the space* c_0.

Theorem 2. *Every absolutely convergent double series converges perfectly. The converse statement is not true even for numerical series: There exists a perfectly convergent numerical double series* $\sum a_{ij}$ *that is not absolutely convergent.*

Theorem 3. *A double series converges unconditionally if and only if it converges perfectly.*

Let us introduce certain notions and consider some auxiliary statements.

For ordinary perfectly convergent series, the following theorem is true:

Gel'fand Theorem [1, p. 9]. *If a series* $\sum x_n$ *converges perfectly, then the series* $\sum \alpha_n x_n$ *converges uniformly on* D, *i.e.,*

$$\forall\,\varepsilon>0 \quad \exists n\in\mathbb{N} \quad \forall\, m>n, \quad \forall\,\alpha\in D \quad \left\|\sum_m \alpha_i x_i\right\| < \varepsilon.$$

In this case, the set of sums $S(\alpha)=\sum \alpha_n x_n$ *is a compact set in* X. *The converse statement is also true.*

Space $\mathrm{Perf}(X)$. The Banach space $\mathrm{Perf}(X)$ of all perfectly convergent series in X is defined as a linear space of all sequences $\bar{x} = \{x_n\}_1^\infty \subset X$ that generate perfectly convergent series with coordinatewise summation and multiplication by scalars. This linear space is equipped with the norm

$$\|\bar{x}\| = \|\{x_n\}_1^\infty\| = \sup\{\|S(\alpha)\|:\ \alpha\in D\}.$$

The Gel'fand theorem can be given in the following equivalent form:

Proposition 2. *A series* $\sum x_n$ *converges perfectly if and only if a linear operator* $A: c_0\to X$ *defined by the equality* $Ae_j=x_j$, $j\in\mathbb{N}$, *is compact. In other words, for any Banach space* X, $\mathrm{Perf}(X) = K(c_0, X)$, *where* $K(c_0, X)$ *is the Banach space of all compact operators.*

Note that the space $\mathrm{Perf}(X)$ can also be represented as the following tensor product [2, p. 148]:

$$\mathrm{Perf}(X) = K(c_0, X) = l_1 \otimes_\varepsilon X.$$

We also give two auxiliary statements.

Lemma 1. (Kalton [3]) *The space* $\mathrm{Perf}(X)$ *contains a subspace isomorphic to* c_0 *if and only if* X *contains such a subspace.*

Lemma 2. *If a Banach space* Y *contains no subspaces isomorphic to* c_0, *then every bounded linear operator* $A: c_0\to Y$ *is compact.*

Proof. Since c_0 is isomorphic to the space of all continuous functions on a one-point compactification of a natural series and Y does not contain c_0, by the Pelczynski theorem [4], the operator A is weakly compact. Since the space c_0 does not contain subspaces isomorphic to l_1, by the Rosenthal theorem [5], every bounded sequence in it contains a weakly fundamental subsequence. According to Theorem 4 in [6, p. 532], every weakly compact operator that acts from c_0 into any Banach space maps weakly fundamental sequences into strongly convergent ones. The argument presented above implies that the operator A maps a unit sphere c_0 into a strongly compact subset of the space Y. Lemma 1 is proved.

Compact set D^2. The set of all pairs of sequences of "signs" $\alpha = (\alpha_i)$ appearing in the definition of perfect convergence turns out to be a metrizable compact set if we define a fundamental system of neighborhoods of every point as follows:

$$O_{mn}(\alpha^0,\beta^0) = \{(\alpha,\beta)\in D^2:\ \alpha_i=\alpha_i^0,\ \beta_j=\beta_j^0 \ \text{for}\ 1\le i\le m,\ 1\le j\le n\}.$$

Every perfectly convergent series $\sum x_{ij}$ can now be associated with the function $S(\alpha,\beta)=\sum \alpha_i\beta_j x_{ij}$ on the compact set D^2 with values in X. In this case, every partial sum $S_{mn}(\alpha,\beta)$ in (1a) is a continuous function. The functions introduced above will be used in the proof of Proposition 3.

Functional $M(A)$. For an $m \times n$ matrix $A_{mn} = (x_{ij})$ formed of elements of a Banach space X, we define the functional

$$M(A_{mn}) = \sup\left\{\left\|\sum \alpha_i \beta_j x_{ij}\right\|: \alpha_i, \beta_j = \pm 1\right\}.$$

For an infinite matrix $A_{mn} = (x_{ij})$ formed of terms of a perfectly convergent series $\sum x_{ij}$, the value of the functional $M(A)$ can be determined by passing to the limit

$$M(A) = \lim_{m,n \to \infty} M(A_{mn}) = \sup\left\{\left\|\sum \alpha_i \beta_j x_{ij}\right\|: (\alpha, \beta) \in D^2\right\}.$$

Below (in the proof of Proposition 3), we show that $M(A) < \infty$ for any matrix A generated by a perfectly convergent series.

Lemma 3. *The functional $M(A)$ is the norm on the space of all perfectly convergent double series. If a row or a column is removed from the matrix A (i.e., the corresponding elements of the matrix are assumed to be equal to zero), the value of the functional does not increase. If we select a submatrix B of the matrix A, then $M(A - B) \le 2M(A)$.*

Proof. The fact that $M(A)$ is a norm can be verified directly. Below (in the proof of Theorem 1), we establish that the space of all perfectly convergent series is complete in the norm $M(A)$. We can remove a column (or a row) from the matrix A by calculating the half sum of the matrix A and the matrix A' derived from A by the multiplication of the elements of this row or column by -1. Since $M(A) = M(A')$, we have $M((A + A')/2) \le M(A)$. Every submatrix B can be obtained from the matrix A by omitting the rows and columns that do not compose the matrix B. Therefore, $M(B) \le M(A)$ and, hence, $M(A - B) \le M(A) + M(B) \le 2M(A)$.

Let us prove a statement that generalizes the Gel'fand theorem to the case of double series.

Proposition 3. *If a series $\sum x_{ij}$ converges perfectly, then the set of sums of the series $\sum \alpha_i \beta_j x_{ij}$, $(\alpha, \beta) \in D^2$, is a compact set in X.*

Proof. It follows from Definition 3 that a sequence of continuous functions (partial sums) $S_{mn}(\alpha, \beta)$ converges pointwise on the compact set D^2 to a function $S(\alpha, \beta)$. This means that $S(\alpha, \beta)$ is the Baire function of the first class and, consequently, it possesses a point of continuity $(\alpha^0, \beta^0) \in D^2$. Given $\varepsilon > 0$, we determine indices m and n so that the function $S(\alpha, \beta)$ varies weakly in the neighborhood $O_{mn}(\alpha^0, \beta^0)$, namely,

$$\| S(\alpha, \beta) - S(\alpha^0, \beta^0) \| < \varepsilon/8 \quad \forall (\alpha, \beta) \in O_{mn}(\alpha^0, \beta^0), \tag{2}$$

and is fairly well approximated by the corresponding partial sum at the point (α^0, β^0), namely,

$$\| S(\alpha^0, \beta^0) - S_{mn}(\alpha^0, \beta^0) \| < \varepsilon/8. \tag{3}$$

Let us fix a point $(\alpha, \beta) \in O_{mn}(\alpha^0, \beta^0)$ and form another point $(\overline{\alpha}, \overline{\beta})$ by changing the signs of the coordinates α_i for $i > m$ and β_j for $j > n$, whereas the other signs remain unchanged. It is clear that $(\overline{\alpha}, \overline{\beta}) \in O_{mn}(\alpha^0, \beta^0)$. Consider the element x defined by the equality

$$x = S(\alpha^0, \beta^0) - [S(\alpha, \beta) + S(\bar{\alpha}, \bar{\beta})]/2,$$

which is equivalent to

$$x = [S(\alpha^0, \beta^0) - S_{mn}(\alpha^0, \beta^0)] + \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \alpha_i \beta_j x_{ij}.$$

It follows from inequalities (2) and (3) that $\|x\| < \varepsilon/8$ and

$$\left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \alpha_i \beta_j x_{ij} \right\| < \varepsilon/4, \tag{4}$$

where, according to the definition of the neighborhood O_{mn}, the coefficients α_i and β_j are arbitrary, $(\alpha, \beta) \in D^2$. By virtue of conditions (b) and (c) of Definition 1, every row series and column series converge perfectly. Therefore, by virtue of the Gel'fand theorem, we get

$$\forall\, \varepsilon > 0, \quad \forall\, m \in \mathbb{N} \quad \exists\, q \in \mathbb{N} \quad \forall\, (\alpha, \beta) \in D^2 : \left\| \sum_{i=1}^{m} \sum_{j=q+1}^{\infty} \alpha_i \beta_j x_{ij} \right\| < \varepsilon/8, \tag{5}$$

$$\forall\, \varepsilon > 0, \quad \forall\, n \in \mathbb{N} \quad \exists\, p \in \mathbb{N} \quad \forall\, (\alpha, \beta) \in D^2 : \left\| \sum_{i=p+1}^{\infty} \sum_{j=1}^{n} \alpha_i \beta_j x_{ij} \right\| < \varepsilon/8. \tag{6}$$

Without loss of generality, we can assume that $p > m$ and $q > n$. By adding inequalities (4), (5), and (6) together, we obtain

$$\sup_{\alpha, \beta} \left\| \left(\sum_{i=1}^{m} \sum_{j=q+1}^{\infty} + \sum_{i=p+1}^{\infty} \sum_{j=1}^{n} + \sum_{i=p+1}^{\infty} \sum_{j=q+1}^{\infty} \right) \alpha_i \beta_j x_{ij} \right\| \le \varepsilon$$

or

$$\sup_{\alpha, \beta} \| S(\alpha, \beta) - S_{pq}(\alpha, \beta) \| \le \varepsilon. \tag{7}$$

Thus, we establish that the sequence of continuous functions $S_{pq}(\alpha, \beta)$ converges uniformly to $S(\alpha, \beta)$ on D^2. This means that the function $S_{pq}(\alpha, \beta)$ is continuous on the compact set D^2 and, hence, the set of its values is a compact set in X.

Let us prove the principal results of the present paper.

Proof of Theorem 1. Let us prove the implication $(A) \Rightarrow (B)$. The norm of an arbitrary linear operator $A : c_0 \to Y$ can be calculated according to the formula

$$\|A\| = \sup_{\beta} \left\| \sum \beta_j A e_j \right\|,$$

where the upper bound is taken over all possible collections of the coefficients $\beta_j = \pm 1$. Let us extend our operator

A, which was defined only on the unit vectors e_j, to linear combinations of the unit vectors and calculate its norm. We have

$$\|A\| = \sup_{\beta}\left\|\sum \beta_j A e_j\right\| = \sup\left\{\left\|\sum \alpha_i \beta_j x_{ij}\right\|: \alpha \in D,\ \beta_j = \pm 1\right\}.$$

By extending the upper bound to all $\beta \in D$, we obtain the estimate $\|A\| \le M(A)$ [below, we establish that $\|A\| = M(A)$]. Let us now prove the compactness of the operator A. Consider the finite-dimensional operators A_{mn}: $c_0 \to \mathrm{Perf}(X)$ defined by the equalities

$$A_{mn}e_j = \{x_{ij}\}_{i=1}^{m} \quad \text{for } 1 \le j \le n,$$

$$A_{mn}e_j = 0 \quad \text{for } j > n.$$

Let us show that these operators approximate the operator A with any degree of accuracy, which implies its compactness. We have

$$\|A - A_{mn}\| \le M(A - A_{mn}) = \sup_{\alpha,\beta}\|S(\alpha,\beta) - S_{mn}(\alpha,\beta)\|.$$

By virtue of inequality (7), the last expression tends to zero as $m, n \to \infty$. By virtue of the compactness of the operator A proved above, we can extend its definition to the space $l_\infty = (c_0)^{**}$, preserving the range of its values, i.e., $A^{**}: l_\infty \to \mathrm{Perf}(X)$. This implies that $\|A\| = M(A)$.

Let us prove the implication $(B) \Rightarrow (A)$. For any $m \in \mathbb{N}$, we form a set $C_m \subset c_0$ by setting

$$C_m = \{\gamma = (\gamma_i) \in c_0 : \gamma_i = 0 \text{ for } 1 \le i \le m,\ \gamma_i = \pm 1 \text{ for } m < i \le m_1 \text{ with arbitrary } m_1,\ \gamma_i = 0 \text{ for } i > m_1\}.$$

It is clear that any sequence of "representatives" $\gamma^{(m)} \in C_m$ weakly converges to zero as $m \to \infty$. Since the operator A is compact, the sequence of images $A\gamma^{(m)}$ strongly converges to zero. This means that, for any $\varepsilon > 0$, there exists m such that the set AC_m (and all sets AC_k for $k > m$) lies in the $\varepsilon/2$-neighborhood of zero in $\mathrm{Perf}(X)$. The same is also true for the weak closure $\overline{C_m}$ of the set C_m in l_∞, i.e.,

$$\sup\{\|A\gamma\| : \gamma \in C_m\} = \sup_{\alpha,\beta}\left\|\sum_{j=1}^{\infty}\beta_j \sum_{i=m+1}^{\infty}\alpha_i x_{ij}\right\| < \varepsilon/2.$$

This, in particular, means that if we remove all rows beginning with the number $i = m + 1$ from the matrix A, we obtain the matrix

$$A_m = \{x_{ij}^{(m)} = x_{ij} \text{ at } 1 \le j \le m,\quad x_{ij}^{(m)} = 0 \text{ at } i > m;\ j \in \mathbb{N}\},$$

which slightly differs from the original matrix: $\|A - A_m\| < \varepsilon/2$. By applying the Gel'fand theorem to finitely many nonzero row series and properly choosing n, we obtain the finite matrix

$$A_{mn} = \{x_{ij}^{(mn)} = x_{ij} \text{ for } 1 \le i \le m,\ 1 \le j \le m,\quad x_{ij}^{(mn)} = 0, \text{ otherwise}\},$$

which slightly differs from the matrix A:

$$\|A - A_{mn}\| < \varepsilon/2. \tag{8}$$

In other words, the double series $\sum \alpha_i \beta_j x_{ij}$ converge in acertain increasing sequence of rectangular partial sums uniformly with respect to $(\alpha, \beta) \in D^2$. Finally, let us show that the series $\sum x_{ij}$ converges perfectly. For given $\varepsilon > 0$, we take m and n such that inequality (8) is satisfied. Now let $p > m$ and $q > n$ be arbitrary. The matrix $A - A_{pq}$ is obtained from the matrix $A - A_{mn}$ by removing the submatrix $A_{pq} - A_{mn}$ According to Lemma 4, we have

$$\|A - A_{pq}\| < \varepsilon$$

for all $p > m$ and $q > n$. By Proposition 1, the series $\sum x_{ij}$ converges perfectly. The implication $(B) \Rightarrow (A)$ is proved.

Corollary 1. *If a Banach space X does not contain subspaces isomorphic to c_0 (in particular, if $X = \mathbb{R}$), then a sufficient condition of the perfect convergence of the series $\sum x_{ij}$ in X is the boundedness of the operator $A: c_0 \to \mathrm{Perf}(X)$ defined in the conditions of Theorem 1.*

Proof. By Lemma 1, the space $\mathrm{Perf}(X)$ does not contain subspaces isomorphic to c_0. By Lemma 2, every bounded operator $A: c_0 \to \mathrm{Perf}(X)$ is compact. It remains to use the implication $(B) \Rightarrow (A)$ from Theorem 1.

Proof of Theorem 2. The first part of the theorem is obvious, namely, the absolute convergence of a series implies all other types of its convergence. To prove the second part of the theorem, we construct the corresponding example. Recall that an orthogonal matrix of order $m = 2^n$ is called a Walsh matrix W_n, $n \in \mathbb{N}$, if all elements of it are equal to ± 1. A sequence of Walsh matrices can be defined by induction as follows:

$$W_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} W_1 & W_1 \\ W_1 & -W_1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} W_2 & W_2 \\ W_2 & -W_2 \end{pmatrix}, \ldots .$$

The sum of the moduli of elements of the Walsh matrix is $N(W_n) = m^2$. Let us find the upper bound for the values of the functional $M(W_n)$. We have

$$M(W_n) = \max\left\{\sum \alpha_i \beta_j w_{ij} : \alpha, \beta \in D\right\} = \max_{\beta} \sum_{i=1}^{m} \left| \sum_{j=1}^{m} \beta_j w_{ij} \right|.$$

By applying the Cauchy inequality to the last expression, we get

$$M(W_n) \le \max_{\beta} \sqrt{m} \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{m} \beta_j w_{ij} \right)^2 \right)^{1/2} = \sqrt{m} \max_{\beta} \left(\sum_{i,j,k} \beta_j \beta_k w_{ij} w_{ik} \right)^{1/2}.$$

In view of the fact that, by definition, any Walsh matrix is orthogonal, we obtain

$$M(W_n) \le \sqrt{m}\max_{\beta}\left(\sum_{j,k}\beta_j\beta_k\left(\sum_i w_{ij}w_{ik}\right)\right)^{1/2} = \sqrt{m}\left(\sum_{i,j} w_{ij}^2\right)^{1/2} = m^{3/2}.$$

We now introduce the matrices $A_n = 2^{-2n}W_n$. It is clear that

$$N(A_n) = 1, \quad M(A_n) \le 2^{-n/2}, \quad n \in \mathbb{N}.$$

Let us form an infinite matrix $A = (a_{ij})$ by inserting the matrices A_n on its diagonal and setting the other elements to be zero. Since $M(A) \le \sqrt{2} + 1$, the double series $\sum a_{ij}$ converges perfectly by Corollary 1. It is not absolutely convergent because $N(A) = \infty$.

Proof of Theorem 3. Let a series $\sum x_{ij}$ converge perfectly. In addition to the compact operator $A: c_0 \to \mathrm{Perf}(X)$ generated by the series $\sum x_{ij}$, for any rearrangement of the natural series σ, we introduce an operator $S: c_0 \to c_0$ that rearranges unit vectors of the space $Se_j = e_{\sigma(j)}$. Furthermore, for some rearrangement π, we introduce an operator P that rearranges the "coordinates" of any element $\bar{x} = \{x_i\}_1^\infty$: $P\bar{x} = \{x_{\pi(i)}\}_1^\infty$. It is clear that S and P are surjective isometries of the corresponding spaces. Then the operator $PAS: c_0 \to \mathrm{Perf}(X)$ is also compact and, hence, the rearranged series $\sum x_{\pi(i),\sigma(j)}$ (perfectly) converges. Since the rearrangements π and σ are arbitrary, the series $\sum x_{ij}$ converges unconditionally. Let us pass to the proof of the converse statement. Let the series $\sum x_{ij}$ be not perfectly convergent. In this case, if a certain column series or row series is not perfectly convergent, then, by virtue of the coincidence of perfect and unconditional convergences of ordinary series, the series $\sum x_{ij}$ is also not unconditionally convergent. Now assume that the series is not perfectly convergent but each column series and row series of it converge perfectly. By virtue of this assumption, we can define the operator A at least on the linear span of unit vectors of the space c_0. Consider the series in $\mathrm{Perf}(X)$ formed of images of the unit vectors from c_0: $\sum Ae_j = \sum \bar{x}_j$. Let this series be perfectly convergent. Then, by the Gel'fand theorem, the sums of the series $\sum \beta_j\bar{x}_j$, $\beta \in D$, form a compact set. Since the image of a unit sphere in the space c_0 is a subset of a closed convex hull of the mentioned compact set, A is a compact operator and, hence, the series $\sum x_{ij}$ is perfectly convergent. Thus, we arrive at a contradiction. Therefore, the series $\sum \bar{x}_j$ is not perfectly convergent. This means that it is also not unconditionally convergent. Therefore, there exists its divergent rearrangement. We preserve the notation $\sum \bar{x}_j$ and $\sum x_{ij}$ for this rearrangement and for the corresponding rearrangement of the double series. Since the series $\sum \bar{x}_j$ diverges, there exists a sequence of its segments

$$\bar{y}_k = \sum_{j=p_k}^{q_k} \bar{x}_j, \quad 1 \le p_1 < q_1 < p_2 < q_2 < \ldots,$$

that is bounded in norm from below by a positive number a, namely,

$$\|\bar{y}_k\| = \sup_{\alpha,\beta}\left\|\sum_{i=1}^{\infty}\sum_{j=p_k}^{q_k}\alpha_i\beta_j x_{ij}\right\| \ge a. \tag{9}$$

Let us perform the following inductive process:

Step 1. Taking into account that all column series converge perfectly, for the first pair of indices (p_1, q_1), we find m_1 such that

$$\sup_{\alpha,\beta}\left\|\sum_{i=m_1+1}^{\infty}\sum_{j=p_1}^{q_1}\alpha_i\beta_j x_{ij}\right\| < a/3.$$

By comparing this inequality with inequality (9), we get

$$\sup_{\alpha,\beta}\left\|\sum_{i=1}^{m_1}\sum_{j=p_1}^{q_1}\alpha_i\beta_j x_{ij}\right\| > 2a/3 > a/3. \tag{10_1}$$

Step 2. In the sequence of pairs of indices (p_k, q_k), we choose a pair [for simplicity, denote it by (p_2, q_2)] such that $p_2 > q_1$ and

$$\sup_{\alpha,\beta}\left\|\sum_{i=1}^{m_1}\sum_{j=p_2}^{q_2}\alpha_i\beta_j x_{ij}\right\| < a/3.$$

By using the fact that all row series converge perfectly, we define an index $m_2 > m_1$ so that

$$\sup_{\alpha,\beta}\left\|\sum_{i=m_2+1}^{\infty}\sum_{j=p_2}^{q_2}\alpha_i\beta_j x_{ij}\right\| < a/3.$$

By comparing the obtained inequalities with inequality (9), we get

$$\sup_{\alpha,\beta}\left\|\sum_{i=m_1+1}^{m_2}\sum_{j=p_2}^{q_2}\alpha_i\beta_j x_{ij}\right\| > a/3. \tag{10_2}$$

Step 3. In the sequence of pairs of indices (p_k, q_k), we choose a pair [for simplicity, denote it by (p_3, q_3)] such that $p_3 > q_2$ and

$$\sup_{\alpha,\beta}\left\|\sum_{i=1}^{m_2}\sum_{j=p_3}^{q_3}\alpha_i\beta_j x_{ij}\right\| < a/3.$$

We define an index $m_3 > m_2$ so that

$$\sup_{\alpha,\beta}\left\|\sum_{i=m_3+1}^{\infty}\sum_{j=p_3}^{q_3}\alpha_i\beta_j x_{ij}\right\| < a/3.$$

By comparing the obtained inequalities with inequality (9), we get

$$\sup_{\alpha,\beta}\left\|\sum_{i=m_2+1}^{m_3}\sum_{j=p_3}^{q_3}\alpha_i\beta_j x_{ij}\right\| > a/3. \tag{10_3}$$

By infinitely continuing this process (using every time the perfect convergence of row series and column series), we obtain the disjunctive sequence of matrices

$$B_k = \{x_{ij} \text{ for } m_{k-1} < i \le m_k \text{ and } p_k \le j \le q_k; \; 0 \text{ for the other pairs of indices } (i, j)\}$$

whose norms admit the uniform lower bound

$$\|B_k\| > a/3.$$

Now let us fix signs $(\alpha^0, \beta^0) \in D^2$ that realize the norms of the matrices B_k,

$$\|B_k\| = \sup_{\alpha,\beta} \left\| \sum_{i=m_{k-1}+1}^{m_k} \sum_{j=p_k}^{q_k} \alpha_i^0 \beta_j^0 x_{ij} \right\| > a/3,$$

and pass to the construction of a divergent rearrangement of the series $\sum x_{ij}$. Every set

$$H_k = \{m_{k-1} < i \le m_k, \; p_k \le j \le q_k\}$$

of pairs of indices (i, j) can be represented as the union of four subsets

$$H_k = H_k(+,+) \cup H_k(+,-) \cup H_k(-,+) \cup H_k(-,-)$$

in accordance with the values of the coefficients α_i^0 and β_j^0. In this case, for $(i, j) \in H_k$, the sum $\sum_{i,j} \alpha_i^0 \beta_j^0 x_{ij}$ can also be represented as a sum of four terms. In norm, one of these terms [denote it by $H_k(\gamma_k, \delta_k)$] is at least a quarter of the norm of the total sum:

$$\left\{ \left\| \sum_{i,j} \alpha_i^0 \beta_j^0 x_{ij} \right\| \text{ for } (i, j) \in H_k(\gamma_k, \delta_k) \right\} > a/12.$$

Finally, let us find a divergent rearrangement of the series $\sum x_{ij}$. For this purpose, we rearrange the indices i in each segment $m_{k-1} < i \le m_k$ so that the indices from $H_k(\gamma_k, \delta_k)$ occupy the first place. We also perform a similar procedure with the index j. Indices that do not belong to the sets H_k remain at their places. We denote by π and σ, respectively, the obtained "block" rearrangements of the natural series and, for simplicity, denote the rearranged series $\sum x_{\pi(i),\sigma(j)}$ by $\sum x_{ij}^*$. This series now contains the sequence of rectangular sums

$$S^{(k)} = \sum x_{ij}^* \quad \text{for} \quad m_{k-1} < i \le r_k \le m_k \quad \text{and} \quad p_k \le j \le s_k \le q_k,$$

the norms of which, as indicated above, are estimated from below by the number $a/12$. Each sum $S^{(k)}$ can be expressed in terms of rectangular partial sums of the rearranged series as follows:

$$S^{(k)} = \left[S_{r_k s_k} - S_{r_k p_k} \right] - \left[S_{m_{k-1} s_k} - S_{m_{k-1} p_k} \right].$$

This means that one of the differences of partial sums is estimated from below by the number $a/24$ for all $k \in \mathbb{N}$. This implies that the rearranged series is divergent. Theorem 3 is proved.

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