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# A SUFFICIENT CONDITION FOR STRONG ALMOST-PERIODICITY OF SCALARLY ALMOST PERIODIC REPRESENTATIONS OF THE GROUP OF REAL NUMBERS

**M. I. Kadets**

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We prove the following theorem: If every separable subspace $Y$ of a Banach space $X$ has a separable weak sequential closure in $Y^{**}$, then every scalarly almost periodic group acting in $X$ is strongly almost periodic.

First, we define the notions appearing in the title.

***Definition 1.*** *A bounded continuous function $f(t)$ defined on the number axis $\mathbb{R}$ and taking values in a Banach space $X$ is called (strongly) almost periodic if the set of its translations $f(t+\tau)$, $\tau \in \mathbb{R}$, is relatively compact in the uniform metric. For $X = \mathbb{C}$, we obtain the definition of an almost periodic Bohr function.*

***Definition 2.*** *A function $f(t)\colon \mathbb{R} \to X$ is called scalarly almost periodic if, for any linear functional $x^* \in X^*$, the scalar function $\langle x^*, f(t)\rangle$ is an almost periodic Bohr function.*

Sometimes, a scalarly almost periodic function is called a weakly almost periodic function. However, in this case, there is a possibility of confusion between this notion and the notion of a weakly almost periodic function in the sense of Eberlein [1].

Every almost periodic function with values in $X$ is scalarly almost periodic. The converse statement is true if and only if the weak convergence and strong convergence of sequences in $X$ coincide (the Schur property) [2].

***Definition 3.*** *A one-parameter strongly continuous group of linear continuous operators $T(t)$ $[t \in \mathbb{R}$, $T(t_1 + t_2) = T(t_1)T(t_2)$, $T(0) = I]$ acting in a Banach space $X$ is called strongly almost periodic if, for any element $x \in X$, the function $T(t)x$ with values in $X$ is almost periodic (or, equivalently, the set of values of this function is relatively compact in $X$). A group $T(t)$ is scalarly almost periodic if, for any $x \in X$ and $x^* \in X^*$, scalar functions $\langle x^*, T(t)x\rangle$ are Bohr functions.*

Generally speaking, the notions given above are defined for an arbitrary topological group $G$ (almost periodic representations of the group $G$ are considered in [3]). But in the present paper, we restrict ourselves to the case $G = \mathbb{R}$.

The following statement was formulated by Lyubich [4]: If $X$ is weakly sequentially complete (i.e., every weak Cauchy sequence is weakly convergent in this space), then every scalarly almost periodic group is strongly almost periodic. If $X = c$ (the Banach space of all convergent number sequences), then there exists a scalarly almost periodic group which is not strongly almost periodic.

There arises the following problem, which is natural for the theory of functions with values in Banach spaces: What maximally wide class of Banach spaces exists for which the first part of the Lyubich statement indicated above holds true?

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The present paper gives a partial answer to this question.

**Theorem 1.** *Let a Banach space* $X$ *possess the following property (S): For every separable subspace* $Y$ *of it, the weak** *sequential closure of* $Y$ *is separable in the second dual space* $Y^{**}$. *Then every scalarly almost periodic group is strongly almost periodic.*

In what follows, we use some well-known facts concerning scalarly almost periodic functions.

**Proposition 1** [5]. *The set of values of every scalarly almost periodic function is separable.*

**Proposition 2** [6]. *The spectrum of a scalarly almost periodic function is at most countable.*

Recall that the union of the spectra of all almost periodic Bohr functions $\langle x^*, f(t)\rangle$, where $x^*$ runs through the dual space $X^*$, is called the spectrum of a scalarly almost periodic function $f\colon \mathbb{R} \to X$.

The countability of the spectrum of scalarly almost periodic functions makes it possible to extend the approach connected with the compactification of the number axis $\mathbb{R}$ from almost periodic Bohr functions to scalarly almost periodic functions. Let $f(t)\colon \mathbb{R} \to X$ be scalarly almost periodic function, and let $\Lambda = \{\lambda_n\}_1^\infty$ be its spectrum. By using this spectrum (more exactly, by using the smallest number module containing the spectrum), it is possible to equip the number axis by topology with respect to which it becomes a precompact metrizable group $G_0$. (If and only if the function $f(t)$ is periodic, points the distances between which are divisible by the period become identical so that the straight line transforms into a circle. In this case, no considerable difficulties appear.) We denote the complement of the group $G_0$ by $G$. It is a compact metric group. The function $f(t)$ on $G_0$ is weakly uniformly continuous (i.e., each scalar function $\langle x^*, f(t)\rangle$, $x^* \in X^*$, is uniformly continuous). The definition of the function $f(t)$ can be extended by continuity to the compact set $G$ (for a predetermined function, we preserve the notation $f(t)$, $t \in G$). On such an extension of the definition, its values on $G \setminus G_0$ may not belong to $X$. They will belong to the weak sequential* closure of the space $X$ in $X^{**}$. Finally, we note that the strong almost periodicity of a function $f(t)$ is equivalent to the strong continuity of extension of its definition on $G$.

***Proof of Theorem 1.*** For an arbitrary element $y_0 \in X$, we construct the scalarly almost periodic function $f(t) = T(t)y_0$. By $Y$, we denote the closed linear hull of the set of its values and, by $E = E(Y) \subset Y^{**}$, denote its weak* sequential closure in $Y^{**}$. In correspondence with the argument presented above, we compactify the group $\mathbb{R}$ by converting it into a precompact metric group with complement $G$. On the extension of the definition to $G$, the function $f(t)$ becomes a weakly* continuous function with values in $E$. The theorem is proved if we establish that the defined function is strongly continuous. Since the considered function is created by the group $T(t)$, it suffices to prove that this predetermined function is strongly continuous at least at one point $t_0 \in G$.

To continue the proof of the theorem, we need certain information on special equivalent norms in separable Banach spaces. Let $E$ be a separable Banach space, and let $\Gamma$ be a closed linear subspace of the dual space $E^*$. By using $\Gamma$, we define the functional

$$p_\Gamma(e) = \sup\left\{\left|\langle e^*, e\rangle\right| : e^* \in \Gamma,\ \|e^*\| \le 1\right\}$$

on $E$. Further, we define the Dixmier characteristic of the subspace $\Gamma$:

$$r(\Gamma) = \inf\{p_\Gamma(e) : e \in E,\ \|e\| = 1\}.$$

A subspace $\Gamma$ is called normalizing if $r(\Gamma) > 0$ and 1-normalizing if $r(\Gamma) = 1$. Note that $r(\Gamma) = 1$ if and only if

$p_\Gamma(e) = \| e \|$ for all $e \in E$. For the properties (and, in particular, equivalent definitions) of the Dixmier characteristic, see [7, pp. 28–44].

**Proposition 3** [8, pp. 176–184]. *Let $E$ be separable Banach space, and let $\Gamma \subset E^*$ be the normalizing subspace. On $E$, there exists an equivalent norm $\| \cdot \|$ with the property $H(\Gamma)$, i.e., the following two conditions are satisfied:*

$(K_1)$ *if* $\langle e^*, e_n \rangle \xrightarrow[n]{} \langle e^*, e \rangle \ \forall e^* \in \Gamma$, *then* $\liminf \| e_n \| \geq \| e \|$;

$(K_2)$ *if, in addition,* $\| e_n \| \to \| e \|$, *then* $\| e_n - e \| \to 0$.

We prove the following strengthened version of Proposition 3:

**Proposition 4.** *Let $E$ be a separable Banach space , and let $\Gamma \subset E^*$ be the normalizing subspace. On $E$, it is possible to introduce an equivalent norm with property $H(\Gamma)$ with respect to which $\Gamma$ is a 1-normalizing subspace.*

***Proof.*** From the subspace $\Gamma$, we extract a separable normalizing subspace $\Gamma_0$. To do this, on a unit sphere of the space $E$, we select a countable dense subset $\{ e_n \}_1^\infty$. For each $e_n$, we choose in $\Gamma$ an element $e_n^*$, $\| e_n^* \| = 1$, $\langle e_n^*, e_n \rangle \geq p_\Gamma(e_n)/2$. The closure of the linear span of the set $\{ e_n^* \}$ gives the required subspace $\Gamma_0 \subset \Gamma$. According to Proposition 3, we introduce an equivalent norm in $E$ with property $H(\Gamma_0)$ and show that, automatically, $r(\Gamma_0) = 1$. This is implied by the following reasoning:

1. Condition $(K_1)$ means that the unit sphere $B(E)$ is sequentially closed in the weak $\Gamma_0$-topology (i.e., in the topology in which neighborhoods of the zero are determined by finite collections of linear functionals from $\Gamma_0$).

2. Since $\Gamma_0$ is a normalizing subspace, then the topological closure of $B(E)$ in the weak $\Gamma_0$-topology is bounded [it lies in the "blown" sphere $r^{-1}(\Gamma_0)B(E)$].

3. Since $\Gamma_0$ is separable, the weak $\Gamma_0$ -topology is metrizable on each bounded subset of the space $E$. In particular, it is metrizable on the sphere $r^{-1}(\Gamma_0)B(E)$.

4. Since the notions of sequential closure and topological closure coincide in a metric space, the sphere $B(E)$ is closed not only sequentially but also topologically, which implies the required equality $r(\Gamma_0) = 1$.

(For more details concerning the presented reasoning, see [7, pp. 28–34].). Since $\Gamma \supset \Gamma_0$, the property $H(\Gamma)$ and equality $r(\Gamma) = 1$ hold in the same norm.

By using Proposition 4, we prove the statement that allows us to complete the proof of the theorem.

**Lemma 1.** *Let $E$ be a separable Banach space, let $\Gamma$ be a normalizing subspace of the dual space $E^*$, and let $G$ be a metric compact set. Let $f(t)\colon G \to E$ be a $\Gamma$-weakly continuous function. Then there exists a point $x_0 \in G$ at which the function $f(t)$ is strongly continuous.*

***Proof.*** Since all notions from the statement of the lemma have the linearly topological (but not metric) nature, it suffices to prove the lemma in any equivalent norm. In accordance with Proposition 4, we introduce an equivalent norm on $E$ with respect to which $E$ possesses the property $H(\Gamma)$ and $\Gamma$ is a 1-normalizing subspace. Consider the scalar function

$$\varphi(t) = \sup\left\{\left|\left\langle e^*, f(t)\right\rangle\right| : e^* \in \Gamma,\ \|e^*\| \leq 1\right\}, \qquad t \in G.$$

Since $r(\Gamma) = 1$, we have $\varphi(t) = \|f(t)\|_E$. Since $G$ is separable, $\varphi(t)$ is a function of the first Baire class. Consequently, in $G$, there exists a dense $G_\delta$ subset of its points of continuity. Assume that $t_0 \in G$ is one of such points. Then, for any sequence $(t_n)_1^\infty$ that converges to $t_0$, we have

$$f(t_n) \xrightarrow{\Gamma\text{-weak}} f(t_0) \quad [\Gamma\text{-weak continuity of } f(t)],$$

$$\|f(t_n)\| \to \|f(t_0)\| \quad [\text{continuity of } \varphi(t) \text{ for } t = t_0].$$

Since $E$ possesses the property $H(\Gamma)$, we have

$$\lim_{n \to \infty} \|f(t_n) - f(t_0)\| = 0.$$

***Completion of the proof of Theorem 1.*** We have a separable Banach space $Y$ and its weak* sequential closure in $Y^{**}$ (denoted it by $E$), which is separable according to the condition of the theorem. As $\Gamma$, we take the image of $Y^*$ under its natural imbedding into $E^*$. In more detail, consider the isometric imbeddings

$$Y \xrightarrow{i} E \xrightarrow{j} Y^{**};$$

By passing to adjoint operators, we get

$$Y^* \xrightarrow{\pi_1} Y^{***} \xrightarrow{j^*} E^* \xrightarrow{i^*} Y^*$$

(here, $\pi_1$ is a canonical imbedding). Set $\Gamma = j^*\pi_1 Y^*$, which is a normalizing subspace in $E^*$. At the beginning of the proof of the theorem, we considered a weakly* continuous function $f(t)$ defined on the metric compact set $G$ and taking values in a separable subspace of the space $Y^{**}$. In other words, we have a $\Gamma$-weakly continuous function $f(t)$ defined on $G$ and taking values in the separable Banach space $E$. Thus, the conditions of Lemma 1 are satisfied, which implies the existence of at least one point of strong continuity of the function $f(t)$. According to the remark made above, the function $f(t)$ is strongly continuous on $G$, which proves the theorem.

The class of Banach spaces with property $(S)$ indicated in the theorem is strongly wider than the class of weakly sequentially complete spaces from the condition of the Lyubich theorem. In particular, it contains quasi-reflexive spaces.. It is reasonable to assume that the maximally wide class of functions for which Theorem 1 is true is formed by spaces that do not contain subspaces isomorphic to the space $c$.

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