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On the Relationship between the Strong and Scalar Almost Periodicity of Banach Representations of the Group of Reals

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We begin with the definition of the notions mentioned in the title.

Definition 1. A bounded continuous function $f(t)$ on the real line $\mathbb{R}$ with range in a Banach space X is said to be (*strongly*) *almost periodic* (*a.p.*) if the set $f(t+\tau)$, $\tau \in \mathbb{R}$, of its translations is precompact in the uniform metric. For $X = \mathbb{C}$ we obtain the (Bochner) definition of a Bohr a.p. function.

Definition 2. A function $f\colon \mathbb{R} \to X$ is said to be *scalar a.p.* if for each linear functional $x^* \in X^*$ the scalar function $\langle x^*, f(t)\rangle$ is a Bohr a.p. function. Sometimes a scalar a.p. function is said to be weakly a.p., however, this can lead to confusion with the Eberlein a.p. functions.

Every a.p. function with range in X is scalar a.p. The converse holds if and only if X possesses the Schur property (the coincidence of the weak and strong convergence of sequences).

Definition 3. A one-parameter strongly continuous group $T(t)$ ($t \in \mathbb{R}$, $T(t_1+t_2) = T(t_1)T(t_2)$, $T(0) = I$) of continuous linear operators acting on a Banach space X is said to be (*strongly*) *a.p.* if for any $x \in X$ the function $T(t)x$ with range in X is a.p. The group $T(t)$ is said to be *scalar a.p.* if for every $x \in X$ and $x^* \in X^*$ the scalar function $\langle x^*, T(t)x\rangle$ is a Bohr a.p. function.

In general, the above notions make sense for an arbitrary topological group G_0 (almost periodic representations of the group G_0; see [1]), but in this note we restrict ourselves to the case $G_0 = \mathbb{R}$. One of the reasons for this restriction is the absence of a theorem stating that the spectrum of a scalar a.p. function on a group distinct from $\mathbb{R}$ is at most countable (see [4]).

The following assertion is due to Yu. I. Lyubich: *if X is weakly sequentially complete, then each scalar a.p. group acting on X is a.p. However, if $X = c$ (c is the Banach space of all convergent number sequences), then there exists a scalar a.p. group that is not a.p.* The following natural question in the theory of functions with ranges in Banach spaces arises: what is the maximal class of Banach spaces for which the affirmative part of the Lyubich assertion holds? This note gives a partial answer to this question.

Theorem. *Let a Banach space X have the following property: the weak* sequential closure of each of its separable subspaces Y in the second conjugate space Y^{**} is separable. Then each scalar a.p. group acting on X is a.p.*

The beginning of the proof of the theorem. Take an arbitrary element $y_0 \in X$ and form the scalar a.p. function $f(t) = T(t)y_0$. Denote by Y the closed linear span of the range of this function. Since the range of a scalar a.p. function is separable [3], Y is separable. Denote by E, $E \subset Y^{**}$, the weak* sequential closure of the subspace Y in Y^{**}. Because the spectrum of the function $f(t)$ is at most countable [4], the compactification of the real axis $\mathbb{R}$ by this spectrum results in a precompact metric group whose completion G is a compact metric group. (The case of a purely periodic function $f(t)$ for which the compactification transforms $\mathbb{R}$ into a circle involves no additional difficulties.) The extension of the function $f(t)$ to G turns out to be a weakly* continuous function with range in E. The theorem will be proved if we establish the strong continuity of the extended function. Since the function under consideration is generated by the group $T(t)$, it suffices to prove its strong continuity at at least one point $t_0 \in G$ (we preserve the notation $f(t)$ for the extended function).

To complete the proof, we need an assertion concerning special equivalent norms in separable Banach spaces. Let E be a separable Banach space and let Γ be a closed linear subspace of the conjugate space E^*. By means of Γ we define on E the seminorm $p(e) = \sup\{|\langle e^*, e\rangle| : e^* \in \Gamma,\ \|e^*\| \le 1\}$.

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The *Dixmier characteristic* of the subspace Γ is defined as follows: $r(\Gamma) = \inf\{p(e) : e \in E, \|e\| = 1\}$. A subspace Γ is said to be *norming* if $r(\Gamma) > 0$; it is said to be *1-norming* if $r(\Gamma) = 1$.

Definition 4. Let E be a Banach space and let Γ be a subspace of E^*. We will say that E *has the property* $H(\Gamma)$ if the following two conditions are satisfied: (K_1) if $\langle e^*, e_n\rangle \to \langle e^*, e\rangle$ for all $e^* \in \Gamma$, then $\liminf \|e_n\| \geq \|e\|$ and (K_2) if, in addition, we have $\|e_n\| \to \|e\|$, then $\|e_n - e\| \to 0$.

Proposition 1. *Let E be a separable Banach space and let $\Gamma \subset E^*$ be a norming subspace. Then on E there exists an equivalent norm with property $H(\Gamma)$ such that Γ is 1-norming with respect to this norm.*

This is a sharpening of the well-known theorem on equivalent norms with property $H(\Gamma)$ (e.g., see [5, p. 176–184]). Proposition 1 makes it possible to prove the following assertion which is necessary to complete the proof of the theorem.

Proposition 2. *Let E be a separable Banach space, let $\Gamma \subset E^*$ be a norming subspace, and let G be a compact metric space. Let $f\colon G \to E$ be a Γ-weakly continuous function. Then there exists a point $t_0 \in G$ at which the function $f(t)$ is strongly continuous.*

The completion of the proof of the theorem. We have a separable Banach space Y and its weak* sequential closure E in Y^{**}, where E is separable by the condition of the theorem. As the norming subspace Γ we take the image of Y^* under its natural embedding in E^*. In the beginning of the proof we had a weakly* continuous function $f(t)$ on a compact metric space G with range in a separable subspace $E \subset Y^{**}$. In other words, we had a Γ-weakly continuous function on G with range in a separable space E. Thus, it turns out that the assumptions of Proposition 2 hold, which guarantees the existence of at least one point of strong continuity of the function $f(t)$. By the above remark on the nature of the function $f(t)$, it turns out to be strongly continuous on G, and this proves the theorem.

It is also possible to consider the problem on the coincidence of the strong and scalar almost periodicity for group representations in arbitrary Banach spaces. Here is a sample result [6]: *In every Banach space a representation that is simultaneously scalar a.p. and weakly (Eberlein) a.p. is a.p.*

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