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RELATION BETWEEN SOME PROPERTIES OF CONVEXITY OF THE UNIT BALL OF A BANACH SPACE

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According to a fundamental theorem of Rosenthal [1], a Banach space does not contain subspaces isomorphic to l_1 if and only if every sequence of elements of the space contains a weak Cauchy subsequence. Rosenthal's theorem implies several topological linear consequences (see, the survey, [2]). In this note we show that the absence of subspaces isomorphic to l_1 may manifest itself on the geometric properties of the space as well.

Let us recall some definitions. A Banach space X is said to be strictly convex (in notation: $X \in R$) if its unit sphere does not contain straight line segments. A Banach space is said to be symmetrically locally uniformly convex (in notation: $X \in MLUR$) if the conditions $\|x_0\| = 1$, $\|x_0 \pm y_n\| \to 1$ as $n \to \infty$ imply that $\|y_n\| \to 0$. A Banach space is said to be locally uniformly convex (in notation: $x \in LUR$) if the conditions $\|x_0\| = 1$, $\|x_0 + y_n\| \to 1$ and $\|x_0 + 1/2 y_n\| \to 1$ as $n \to \infty$ imply that $\|y_n\| \to 0$. A Banach space has the property H (in notation: $X \in H$) if the conditions $\|x_0\| = 1$, $\|x_0 + y_n\| \to 1$ and $y_n \xrightarrow{w} 0$ as $n \to \infty$ imply that $\|y_n\| \to 0$. The property MLUR (the least known among those mentioned above) was introduced by Anderson in [3]; here we have given an equivalent definition of this property, proposed by Smith and Turett [4].

The following implications are known:

$$(X \in LUR) \Rightarrow (X \in MLUR) \Rightarrow (X \in R), \quad (X \in LUR) \Rightarrow (X \in H). \tag{1}$$

The following question was raised in [3]: under what conditions on X does the implication

$$(X \in R) \text{ and } (X \in H) \to (X \in MLUR) \tag{2}$$

hold? Smith [5] has constructed an example (a space isomorphic to l_1) for which implication (2) does not hold. In [4] it is conjectured that implication (2) holds if X* satisfies the Radon–Nikodym condition. We are now going to prove a stronger statement.

THEOREM 1. Let X not contain subspaces isomorphic to l_1. If $X \in R$ and $X \in H$, then $X \in MLUR$.

Proof. Let us assume the contrary: $X \notin MLUR$. Then there exist a normalized element $x_0 \in X$ and a sequence $(y_n) \subset X$ such that

$$\lim_{n\to\infty} \|x_0 \pm y_n\| = \|x_0\| = 1, \quad \text{but} \quad \|y_n\| \geqslant \delta > 0. \tag{3}$$

Since X does not contain subspaces isomorphic to l_1, without loss of generality we may assume that (y_n) is a weak Cauchy sequence. Let us form the sequence of the elements $x_0 + {}^1/_2 (y_p - y_q)$, where p and q increase unboundedly. This sequence converges weakly to x_0. Let us consider the triangle inequality

$$\left\| x_0 \pm \frac{1}{2}(y_p - y_q) \right\| \leqslant \frac{1}{2} [\|x_0 \pm y_p\| + \|x_0 \mp y_q\|] \tag{4}$$

and pass to the limit $p \to \infty$ and $q \to \infty$. By (3), the right-hand side gives $\|x_0\|$. For the left-hand side we obtain

$$\underline{\lim} \left\| x_0 \pm \frac{1}{2}(y_p - y_q) \right\| \geqslant \|x_0\|,$$

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since the sequence $y_p - y_q$ converges weakly to zero. Comparing the limits of the left and right sides of inequality (4), we obtain

$$\lim \left\| x_0 + \frac{1}{2}(y_p - y_q) \right\| = \| x_0 \|.$$

Since by the hypothesis of the theorem, X has the property H, we have $\| y_p - y_q \| \to 0$, and conversely, (y_n) converges strongly to some element y, $\|y\| \geq \delta > 0$. From the limit relation (3) we obtain $\|x_0 \pm y\| = \|x_0\|$, which contradicts the strict convexity of X. We have arrived at a contradiction to one of the hypotheses of the theorem.

The presence of the Radon–Nikodym property in X* implies the absence, in X, of subspaces isomorphic to l_1; the converse is not true. Therefore, the statement proved above is stronger than the conjecture in [4].

Below we shall prove that Theorem 1 cannot be strengthened in the class of separable Banach spaces. First of all, we recall an example of Smith [5].

Example. Let $(e_n)_0^\infty$ be the canonical basis in l_1. First let us introduce the equivalent norm

$$\|x\| = \max\left\{|a_0|, \sum_1^\infty |a_n|\right\} \qquad \left(x = \sum_0^\infty a_n e_n\right). \tag{5}$$

in l_1. We denote by E the space l_1 equipped with this norm. We shall need it in what follows. The desired norm is defined by the equality

$$\|x\|_1 = \|x\| + \sqrt{\sum_0^\infty c_n \cdot |a_n|^2}, \tag{6}$$

where (c_n) is a fixed numerical sequence, $c_n > 0$, $c_n \to 0$. The space l_1 equipped with any equivalent norm has the property H, since strong convergence coincides with weak convergence in it (the Schur property). The second term in (6) guarantees the strict convexity of the space. On the other hand, it is easy to verify that

$$\lim_{n\to\infty} \|e_0 \pm e_n\|_1 = \|e_0\|_1, \qquad \|e_n\|_1 > 1, \tag{7}$$

from which it follows that the space E_1 thus constructed is not MLUR.

LEMMA. Let X be a Banach space and let Y be a subspace of X, isomorphic to l_1. Moreover, assume that in the quotient space X/Y there exists an equivalent norm having the property H. Let us denote by p(x) the seminorm induced in X by this equivalent. Then the conditions

$$x_n \xrightarrow{W} x_0, \qquad p(x_n) \to p(x_0) \tag{8}$$

imply that $\|x_n - x_0\| \to 0$.

Proof. Let us denote by $F: X \to X/Y$ the canonical mapping of X onto X/Y. In the quotient space, conditions (8) induce the relations

$$Fx_n \xrightarrow{W} Fx_0, \qquad \|Fx_n\| \to \|Fx_0\|, \tag{9}$$

where $\|Fx\| \equiv p(x)$ is a norm with the property H, whose existence is ensured by the hypotheses of the lemma. From (9) it follows that $\|Fx_n - Fx_0\| \to 0$. In X this means that (x_n) gets arbitrarily close to the set $x_0 + Y$. In other words,

$$x_n = x_0 + y_n + z_n, \text{ where } \quad y_n \in Y, \quad \|z_n\| \to 0. \tag{10}$$

From (8) and (10) it follows that (y_n) converges weakly to zero and by the Schur property, $\|y_n\| \to 0$. Therefore, $\|x_n - x_0\| \to 0$.

THEOREM 2. Let X be a separable Banach space containing a subspace Y isomorphic to l_1. Then it is possible to introduce an equivalent norm in X with respect to which it will become strictly convex, it will have the property H but it will not be MLUR.

Proof. Let $T: Y \to E$ be an isomorphism of Y onto the space E constructed in the example and let $\|T\| < 1$. In X we introduce the equivalent norm

$$\|x\|_1 = \inf_{y\in Y} [\|x - y\| + \|Ty\|], \qquad 1 \leqslant \|x\|/\|x\|_1 \leqslant \|T^{-1}\|. \tag{11}$$

Geometrically, this means that the new unit ball is the closed convex hull of the old ball of X and the set $T^{-1}(V)$, where V is the unit ball of E. We shall need the following property of the new norm: if $x \in Y$ then $\|x\|_1 = \|Tx\|$. Moreover, in X/Y we introduce an equivalent norm with the property H; this is possible in view of the separability of X (cf. [6]). The corresponding seminorm in X will be denoted by p(x), as in the lemma. Finally, in X we choose a minimal system $(x_n)_{-\infty}^{\infty}$ with total conjugate which is an "extension" of the system $(T^{-1}e_n)_0^{\infty}$ ($e_n \in E$ is from the example and $x_n = T^{-1}e_n$ for n = 0, 1, 2, . . .). The existence of such an extension is proved in [7].

Now let $x \in X$. The expansion of x in the system (x_n) has the form $x \sim \sum_{-\infty}^{\infty} f_n(x) x_n$. Let us take a sequence of positive coefficients $(c_n)_{-\infty}^{\infty}$ with the properties $\sum c_n \cdot \|f_n\|^2 < \infty$, $c_n \to 0$. We define the desired equivalent norm in the following way:

$$\|x\|_2 = \|x\|_1 + p(x) + \left[\sum_{-\infty}^{\infty} c_n \cdot |f_n(x)|^2\right]^{1/2}. \tag{12}$$

The strict convexity of X in this norm is guaranteed by the last term; the presence of the property H is guaranteed by the second term (via the lemma). Finally, the absence of the property MLUR can be established in the same way as in the example (here the above-mentioned property of the norm $\|\cdot\|_1$ is used: $\|x\|_1 = \|Tx\|$ for all $x \in Y$).

It is not clear how to extend Theorem 2 to nonseparable Banach spaces.

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OPERATORS IN $C^{(1)}$, INDUCED BY SMOOTH MAPPINGS

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The study of spectral properties of weighted substitution operators $Tf = M(f \circ \varphi)$ is connected with a large number of areas of analysis and has various applications (see, e.g., [1] and the literature cited therein). In particular, it has recently been observed by A. B. Antonevich (oral communication on the seminar LOMI) that the description of spectra of invertible weighted substitution operators in the space $C^{(1)}[0, 1]$ enables us to solve the problem of existence of smooth solutions for some partial differential equations. In [2] such a description is carried out in the more general situation where instead of the interval [0, 1], an n-dimensional $C^{(1)}$-manifold K is considered. The spectrum problem of an operator MT_{φ} in $C^{(1)}(K)$, where φ is a noninvertible smooth substitution, is more complex. The main tool for its solution is the generalization of the Kamowitz–Scheinber theorem [3] to the case of weighted endomorphisms in commutative semisimple Banach algebras, which may be interesting in its own right.

The goal of this note is an exposition of the results obtained in this direction. For the brevity of formulation of the results we assume that all weighted substitution operators considered below are noninvertible.

THEOREM 1. Let K be an n-dimensional $C^{(1)}$-manifold, let φ be a $C^{(1)}$-mapping of K into itself and let $M \in C^{(1)}(K)$. Assume that the φ-periodic points form a set of the first category in K. Then the Fredholm spectrum of the operator T, $T = MT_{\varphi}$ is a disk with center at zero.

THEOREM 2. Let the hypotheses of Theorem 1 be satisfied and let λ be an isolated point of $\sigma(T)$.

Then there exists a φ-periodic attracting point x of period k such that $\lambda^k = M(x), \ldots M(\varphi_{k-1}(x))$. In particular, if φ does not have attracting periodic points, then the spectrum of T coincides with its Fredholm spectrum.

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