Note Added in Proof. After this article was sent to press, a letter from Professor Atiyah informed us that analogous results were obtained independently of us by Atiyah and Hitchin was the aid of the Horrocks—Barth technique.

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THE SUPERREFLEXIVITY PROPERTY OF A BANACH SPACE IN TERMS OF THE CLOSENESS OF ITS FINITE-DIMENSIONAL SUBSPACES TO EUCLIDEAN SPACES

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For a Banach space X and a natural number n we define the functional

$$d_2(X,\ n) = \sup \{d(X_n,\ l_2^{(n)}) : X_n \subset X,\ \dim X_n = n\}.$$

In this note we prove the following theorem.

THEOREM. Let X be a real Banach space. If for some natural number n

$$d_2(X,\ n) < M_n \equiv \left(\sum_{\nu=1}^{n} \sin^{-1}\frac{\pi}{n}\left(\nu-\frac{1}{2}\right)\right) \Big/ n \sim \frac{2}{\pi}\ln n,$$

then X is superreflexive.

According to the James—Schäffer criterion [1] (see also [2]), a space is not superreflexive if and only if for any n and ε there is a collection $\{e_k\}_1^n$ of normalized vectors in it such that

$$\left\|-\sum_{k=1}^{m} e_k + \sum_{m+1}^{n} e_k\right\| \geqslant n(1-\varepsilon) \qquad (0 \leqslant m < n). \tag{J}$$

To simplify the calculations we introduce the "idealized condition J": for any n there is a collection of normalized vectors such that

$$\left\|-\sum_{k=1}^{m} e_k + \sum_{m+1}^{n} e_k\right\| = n \qquad (0 \leqslant m < n). \tag{J_0}$$

The condition J_0 can be given the equivalent form

$$\left\|-\sum_{k=1}^{m} \lambda_k e_k + \sum_{m+1}^{n} \lambda_k e_k\right\| = 1 \qquad \left(\lambda_k \geqslant 0,\ \sum \lambda_k = 1,\ 0 \leqslant m < n\right). \tag{J_1}$$

In other words, a Banach space is not superreflexive if and only if for any n there is an n-dimensional subspace E of it whose norm (to within ε) coincides with the l_1-norm on those vectors $x = \Sigma a_k e_k$, whose coordinates admit not more than one change of sign.

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LEMMA. If a basis $\{e_k\}_n^1$ ($\|e_k\| = 1$) of an n-dimensional real normed space E satisfies the condition J_1, then $d_2(E, l_2^{(n)}) \geqslant M_n$, and the equality sign can be attained.

Proof. In E we introduce the coordinatewise scalar product. Define in E a linear operator T acting according to the equations $Te_k = e_{k+1}$ for $1 \leqslant k < n$ and $Te_n = -e_1$. With the aid of T the condition J_1 can be rewritten as follows:

$$\|T^m x\| = 1 \text{ for } 0 \leqslant m < n \text{ and } x \in \Omega_0 \equiv \{x \in E, (x, e_k) \geqslant 0, \ \Sigma (x, e_k) = 1\}.$$

We define some more sets in E: the (n − 1)-dimensional simplexes Ω_m and their union

$$\Omega_m = T^m \Omega_0 \ (0 \leqslant m < 2n), \ \Omega = \bigcup \Omega_m,$$

the parallelepiped

$$\Pi = \left\{x \in E, \ |(x, T^m e)| \leqslant 1, \ 0 \leqslant m < n, \ e = \sum_1^n e_k\right\}$$

and its faces

$$\Pi_m = T^m \Pi_0, \ \Pi_0 = \{x \in \Pi, (x, e) = 1\}, \ \partial\Pi = \bigcup \Pi_m.$$

It is not hard to show that $T\Omega = \Omega$, $T\Pi = \Pi$, and that each simplex Ω_m lies on the corresponding face Π_m of the parallelepiped Π.

The (Banach–Mazur) distance from E to $l_2^{(n)}$ can be determined as

$$d(E, l_2^{(n)}) = \min_Q \{\max [\sqrt{Q(x)}: x \in S]/\min [\sqrt{Q(x)}: x \in S]\},$$

where Q runs over all positive-definite quadratic forms on E, and S is the unit sphere of E. Since, by the condition J_1, the unit ball of E contains the simplexes Ω_m on its sphere and is itself contained in the parallelepiped Π, the distance can be estimated from below as follows:

$$d(E, l_2^{(n)}) \geqslant \inf_Q \sqrt{\varphi(Q)}, \text{ where } \varphi(Q) = \max_{x \in \Omega} Q(x) / \min_{x \in \partial\Pi} Q(x). \tag{1}$$

The functional φ, which is defined on the set of all positive-definite forms, satisfies the relations

$$\varphi(\lambda Q) = \varphi(Q) \ (\lambda > 0), \ \varphi(Q_1 + Q_2) \leqslant \max\{\varphi(Q_1), \ \varphi(Q_2)\}. \tag{2}$$

Suppose that φ attains a lower bound on the forms Q_1 and Q_2. Then, by (2), it also attains its lower bound on any form $Q = \lambda Q_1 + (1-\lambda) Q_2$ $(0 < \lambda < 1)$. Since the sets Ω and $\partial\Pi$ appearing in (1) are invariant with respect to T, φ takes one and the same value on all the forms $Q_m(x) = Q(T^m x)$ $(0 \leqslant m < n)$. By the properties mentioned for the functional φ and the sets Ω and $\partial\Pi$, for the greatest lower bound in (1) we can restrict our search to the forms that are invariant under T: $Q(x) = Q(Tx)$. Moreover, we can let x run over only Ω_0 and Π_0, and not over the sets Ω and $\partial\Pi$. Finally, we can require that $Q(e_1) = 1$, from which it follows that $Q(e_k) = 1$ $(1 \leqslant k \leqslant n)$.

Each form satisfying these conditions can be represented in the form $Q(x) = (Bx, x)$, where B is a symmetric operator with positive spectrum that commutes with T and is normalized by the condition $(Be_k, e_k) = 1$ $(1 \leqslant k \leqslant n)$. Since the operator T (more precisely, its extension to the natural complexification $\tilde{E}$ of the space E) is, with respect to the coordinatewise scalar product, an isometry with simple spectrum $\lambda_\nu = \exp(\pi i (2\nu - 1)/n)$ and eigenvectors $h_\nu = \left(\sum_k \lambda_\nu^{-k} e_k\right) / \sqrt{n}$ $(1 \leqslant \nu < n)$, the operator B can be represented in the form

$$Bx = \sum_{\nu=1}^n b_\nu \cdot (x, h_\nu) \cdot h_\nu \quad \left(b_\nu > 0, \ \sum b_\nu = n, \ b_\nu = b_{n-\nu+1}\right). \tag{3}$$

Accordingly, we need to calculate

$$M^2 = \inf_B \{\max [(Bx, x): x \in \Omega_0]/\min [(Bx, x): x \in \Pi_0]\},$$

where B runs through the operators of the form (3). By the convexity of the form (Bx, x), its maximum is attained on the boundary points, i.e., on the unit vectors e_k, and, hence, is equal to one. Therefore,

$$M^2 = \inf_{x \in \Pi_0} [\min (Bx, x)]^{-1}.$$

In the usual way we get that the minimum of the quadratic form (Bx, x) on the plane $\{x \in E : (x, e) = 1\}$ is attained at the point $x_0 = B^{-1}e / (B^{-1} e, e)$, which, as is not hard to show, belongs to Π_0. Thus,

$$M^2 = \inf_B (Bx_0, x_0)^{-1} = \inf_B (B^{-1}e, e) = \inf \left\{ \sum_{\nu=1}^{n} b_\nu^{-1} |(h_\nu, e)|^2 : b_\nu > 0, \sum b_\nu = n \right\}. \tag{4}$$

This greatest lower bound is attained for

$$b_\nu = n |(h_\nu, e)| \left(\sum_\mu |(h_\mu, e)| \right)^{-1}. \tag{5}$$

From (4) and (5) we get that

$$M_n = \frac{1}{\sqrt{n}} \sum_\nu |(h_\nu, e)| = \frac{1}{n} \sum_\nu \sin^{-1} \frac{\pi}{n} \left(\nu - \frac{1}{2}\right).$$

It remains to prove the existence of a space E_n for which $d(E_n, l_2^{(n)}) = M_n$. We get such a space if we take the unit ball to be the convex hull of the octahedron $V = \{x \in E : \sum |(x, e_k)| \leqslant 1\}$ and the ellipsoid $E = \{x \in E : (B_0 x, x) \leqslant M_n^{-2}\}$, where B_0 is an operator optimizing (4).

The proof of the theorem follows immediately from the lemma.

COROLLARY. If $\sup d(X_3, l_1^{(3)}) < M_3 = 5/3$, for all three-dimensional subspaces X_3 of X, then X is superreflexive.

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ALGEBRAS OF CONTINUOUS FUNCTIONS ON LOCALLY CONNECTED COMPACT SPACES

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1. Let X be a locally connected compact space, and A a closed subalgebra of the algebra C(X) of all continuous complex functions on X. It was shown in [1] that if A separates points, contains the constants, and $A = A^2$ (i.e., each element in A has the form $g^2, g \in A$), then A = C(X). This general fact turns out to be useful in certain problems of multidimensional holomorphic approximation. Generalizations are given in [2, 3]. In this note we describe some further advances.

Everywhere in the following article the compact set X is assumed to be locally connected. The algebra A is assumed to have a unit. For an algebra with unit we denote by A^{-1} the group of invertible elements, by M(A) the maximal ideal space, by Γ(A) the Shilov boundary, and by $\Gamma_0(A)$ the Choquet boundary. The expression $uf \in A^n$ means that $uf = g^n$ for some function $g \in A$ and some function u such that $|u| = 1$ and u is constant on the connected components of the complement to the zero set of f. If this is true for all $f \in A$, then we write $uA = A^n$.

2. The following lemmas are established by the scheme described in [2].

LEMMA 1. If $uA = A^n$ for some fixed $n \geqslant 2$, then the algebra A is symmetric.

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