NOTE ON THE GAP BETWEEN SUBSPACES

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Let G and H be subspaces of Banach space X. As a measure of their separation from each other the concept of a gap $\theta(G, H)$ (see [1] and [2]) was introduced, and later there were modifications $\tilde{\theta}(G, H)$ (see [3]) and $\hat{\theta}(G, H)$ (see [4]). In certain circumstances, the latter modification is the most suitable. It is defined by the formula

$$\hat{\theta}(G, H) = \max\{\sup_g \rho(g, B_H);\ \sup_h \rho(h, B_G)\}, \tag{1}$$

where g and h run through the unit balls B_G and B_H of the corresponding spaces. The gap $\hat{\theta}$ (and also $\tilde{\theta}$) is a metric on the set of all subspaces. If G and H are finite-dimensional and are of different dimensions, then $\hat{\theta}(G, H) = \theta(G, H) = 1$. All three gaps are equivalent in the sense that they define the same topology on the set of all subspaces.

In certain cases, smallness of the gap implies isomorphism of the subspaces ([2], Theorem 1.2; [4], Theorem 4). We will show that in general this is not the case. To be precise, we will construct a Banach space F, a subspace $H \subset F$ which is isometric to l_2, and a sequence of subspaces $G_n \subset F$ which are isometric to l_{p_n} $(p_n \nearrow 2)$ such that $\theta(G_n; H) \to 0$ (although the spaces l_p are not isomorphic to each other).

We fix $p \in (1, 2)$ and define nonlinear operators A and B acting from l_2 into l_p and l_q, respectively $(p^{-1} + q^{-1} = 1)$: if $x = \{x_i\} \in l_2$, then

$$Ax = \{|x_i|^{2/p} \cdot \operatorname{sign} x_i\}, \qquad Bx = \{|x_i|^{2/q} \cdot \operatorname{sign} x_i\}. \tag{2}$$

These operators effect a homeomorphism between the unit spheres in the spaces l_2, l_p, and l_q. We define a Banach space F_1 as the direct product of the spaces l_p and l_2, using the norm

$$\|(g, h)\| = \sup_y |\langle g, By\rangle + \langle h, y\rangle| \quad (g \in l_p;\ h \in l_2,\ y \in l_2;\ y \in l_2;\ \|y\| = 1). \tag{3}$$

(The symbol $\langle u, v\rangle$ denotes the value of the linear functional v on an element u.) It is not hard to see that expression (3) is a norm with respect to which F_1 is a Banach space, and the subspaces G_1 and H_1 consisting of elements of the form (g, 0) and (0, h), respectively, are isometric to l_p and l_2.

We estimate the gap between G_1 and H_1. It follows immediately from the definition that

$$\hat{\theta}(G_1, H_1) \leqslant \sup_x \|(Ax, 0) - (0, x)\| = \sup_x \|(Ax, -x)\| = \sup_{x, y} |\langle Ax, By\rangle - \langle x, y\rangle| \quad (x, y \in l_2, \|x\| = \|y\| = 1). \tag{4}$$

Thus, we need an upper bound for the expression

$$R = |\langle Ax, By\rangle - \langle x, y\rangle| = \Big|\sum_i |x_i|^{2p} \cdot \operatorname{sign} x_i \cdot |y_i|^{2/q} \cdot \operatorname{sign} y_i - x_i y_i\Big|. \tag{5}$$

If we note that it suffices to consider only elements x and y with nonnegative coordinates, we can rewrite (5) in the form

$$R = \sum_i x_i y_i^{1-\varepsilon} |x_i^{\varepsilon} - y_i^{\varepsilon}| \qquad \left(x_i \geqslant 0,\ y_i \geqslant 0,\ \varepsilon = 1 - \frac{2}{q}\right). \tag{6}$$

From the Lagrange formula and then the Cauchy inequality we obtain

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$$R \leqslant \sum_i x_i y_i^{1-\varepsilon} \varepsilon \max\{x_i^{\varepsilon-1}, y_i^{\varepsilon-1}\} \cdot |x_i - y_i| = \varepsilon \cdot \sum_i \max\{x_i^{\varepsilon} y_i^{1-\varepsilon}, x_i\} \cdot |x_i - y_i| \leqslant$$
$$\leqslant \varepsilon \sum_i \max\{x_i, y_i\} \cdot |x_i - y_i| \leqslant \varepsilon \sum_i (x_i + y_i) \cdot |x_i - y_i| \leqslant \varepsilon \|x + y\| \cdot \|x - y\| \leqslant 2\varepsilon. \tag{7}$$

Thus, the gap between G_1 and H_1 does not exceed $2\varepsilon = 2(2-p)p^{-1}$. If we take $p > 4/3$, we obtain an example of two nonisomorphic subspaces for which the gap between them is less than unity.

We now construct the space F. We pick a sequence

$$1 < p_1 < p_2 < \ldots, \qquad \lim p_n = 2, \tag{8}$$

and for each p_n we construct, as above, a space F_n and in this space we select subspaces G_n and H_n. We establish isometries $T_n: H_n \to l_2$ between the subspaces H_n and the space l_2. We form the product $E = \{F_1 \times F_2 \times \ldots\}_{l_1}$; in this product we select the subspace E_0 consisting of sequences $h = \{h_1, h_2, \ldots\}$ ($h_n \in H_n \subset F_n$), subject to the condition $\Sigma T_n h_n = 0$. We let $F = E/E_0$. It is easy to show that F contains isometric images of the spaces F_n (we denote them by the same symbol F_n) and that all of the subspaces H_n are "pasted together" in F into one subspace H which is isometric to l_2 (cf. [5], the lemma on combining imbeddings). It is clear that F has the required property, namely, it contains a sequence of mutually nonisomorphic subspaces G_n which converge in the gap sense to the subspace H.

In addition to estimate (7), we note that in an arbitrary larger space the gap between subspaces G and H, which are isometric to l_p and l_2 ($p < 2$), respectively, is bounded from below:

$$\hat{\theta}(G, H) \geqslant \frac{1}{2}(\sqrt[p]{2} - \sqrt{2}). \tag{9}$$

To see this, we select unit vectors e_1 and e_2 in G. In the unit ball of H we pick elements x and y which are closest to these unit vectors:

$$\|x - e_1\| \leqslant \hat{\theta}(G, H), \quad \|y - e_2\| \leqslant \hat{\theta}(G, H), \quad \|x\| \leqslant 1, \quad \|y\| \leqslant 1. \tag{10}$$

We estimate $\|x + y\|$ and $\|x-y\|$:

$$\|x \pm y\| \geqslant \|e_1 \pm e_2\| - \|x - e_1\| - \|y - e_2\| \geqslant \sqrt[p]{2} - 2\hat{\theta}(G, H). \tag{11}$$

These inequalities together with the parallelogram law give us

$$2\left(\sqrt[p]{2} - 2\hat{\theta}(G, H)\right)^2 \leqslant \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \leqslant 4,$$

which implies (9).

The above constructions evidently justify the introduction of the following concepts. Let X and Y be arbitrary Banach spaces. By the measure of their proximity $p_0(X, Y)$ ($p_1(X, Y)$) we mean the number

$$p_i(X, Y) = \inf_E \inf_{U, V} \theta(UX, VY) \qquad (i = 0, 1),$$

where E runs through all Banach spaces which contain subspaces which are isometric (isomorphic) to X and Y, and U and V run through all isometric (isomorphic) imbeddings of X and Y into E.

LITERATURE CITED

1. M. G. Krein, M. A. Krasnosel'skii, and D. P. Mil'man, Sb. Trudov In-ta Matem. Akad. Nauk UkrSSR, No. 11, 97-112 (1948).
2. I. Ts. Gokhberg and M. G. Krein, Usp. Matem. Nauk, 12, No. 2, 43-118 (1957).
3. I. Ts. Gokhberg and A. S. Markus, Usp. Matem. Nauk, 14, No. 5, 135-140 (1959).
4. V. I. Gurarii, Teor. Funktsii, Funktsion. Analiz i Ikh Prilozhen., No. 1, 194-204 (1965).
5. V. I. Gurarii, Sibirsk. Matem. Zh., 7, No. 5, 1102-1013 (1966).