ON THE INTEGRATION OF ALMOST PERIODIC FUNCTIONS WITH VALUES IN A BANACH SPACE

M. I. Kadets

A function $x(t)$ $(-\infty < t < \infty)$ with values in a Banach space E is said to be (strongly) almost periodic if the set of its translates $x_\tau(t) = x(t+\tau)$ is relatively compact in the metric $\rho(x, y) = \sup_t \| x(t) - y(t) \|$.

Many of the results concerning numerical almost periodic functions carry over to abstract functions [1], [2]. One of the exceptions is the integration theorem or Bohl-Bohr theorem (see [3], p. 29): if the indefinite integral of a numerical almost periodic function is bounded, then it is also an almost periodic function.

We will say that a Banach space E has the Bohl-Bohr property if, for each almost periodic function x(t) with values in E, the boundedness of the integral

$$X(t) = \int_0^t x(\eta)\, d\eta \tag{1}$$

implies that it is almost periodic.

It is well-known that the space c (the space of all convergent numerical sequences) does not have the Bohl-Bohr property [2]. Let us cite an appropriate example:

$$x(t) = \left\{\frac{1}{2^n} \cos\frac{t}{2^n}\right\}_{n=1}^{\infty}. \tag{2}$$

The integral of this function $X(t) = \{\sin(t/2^n)\}_1^\infty$ is bounded but it is not an almost periodic function. It is obvious that any space that contains a subspace isomorphic to c does not have the Bohl-Bohr property. In [2] and [4] the Bohl-Bohr property was established for certain classes of Banach spaces. The following theorem, stated in [4] as a conjecture, gives a decisive solution of this problem.

THEOREM 1. A Banach space has the Bohl-Bohr property if and only if it does not contain a subspace isomorphic to the space c.

Let us note that our proof does not rely on the Bohl-Bohr theorem so that the latter turns out to be a corollary to Theorem 1.

In connection with Theorem 1 there arises the question as to the restrictions that must be imposed on the integral X(t) in an arbitrary Banach space for it to be an almost periodic function. It is already known [1] that for this it is sufficient to require that the set of values of X(t) be strongly relatively compact. We somewhat strengthen this assertion.

THEOREM 2. If x(t) is an almost periodic function and the set of values of the integral is weakly relatively compact, then the integral X(t) is an almost periodic function.

Here we give a simpler proof than the one outlined in [4].

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We make an obvious remark in order to facilitate later arguments. Let M and N be metric spaces and let M_1 be a dense subset of M. Next let f be a mapping of M_1 into N. For each $a \in M$ we can define the oscillation of the function f:

$$\operatorname{osc}(f, a, \varepsilon) = \sup \rho(f(a'), f(a'')),$$

where the upper bound is taken over all $a', a'' \in M_1$ such that $\rho(a^i, a) < \varepsilon$. Thus we can speak of the continuity (or discontinuity) of the function at the point a although the function is not defined at this point.

Let x(t) be an almost periodic function with values in E. We can use this function to introduce a new metric $\rho(t', t'') = \sup_t \|x(t+t') - x(t+t'')\|$ on the real axis; with this metric, the axis becomes a metric space J whose completion K is compact.

It is well known that each function (numerical or abstract) defined on J and continuous at each point of K is almost periodic if we regard it as a function defined on the axis.

We can regard the integral (1) of x(t) as a function defined on J whose value lies in E.

LEMMA 1. If the integral X(t) is continuous at some point $\varkappa_0$ of the compact space K then it is continuous at each point of K.

Proof. Given an arbitrary $\varepsilon > 0$ we define a neighborhood of the point $\varkappa_0$ in which the oscillation of X(t) is less than $\varepsilon/4$; let δ denote the radius of this neighborhood. We select a sequence $\{t^{(n)}\}_{-\infty}^{+\infty}$, relatively dense on the axis, such that $\rho(t^{(n)}, \varkappa_0) < \delta/2$, $0 < t^{(n+1)} - t^{(n)} < l = l(\delta)$. The existence of such a sequence is guaranteed by the almost periodic character of x(t). Let t_0 be an arbitrary point of J. We are going to show that, in the neighborhood of t_0 with radius $\delta_1 = \min\{\delta/2; \varepsilon/4l\}$, the oscillation of the function X(t) is less than ε. In our selected sequence we take a point $t^{(m)}$ for which $t^{(m)} \le t_0 < t^{(m+1)}$. We consider the identity

$$X(t) - X(t_0) = \{X(t^{(m)} + \tau) - X(t^{(m)})\} + \int_{t^{(m)}}^{t_0} [x(\eta + \tau) - x(\eta)]\, d\eta, \tag{3}$$

where t is an arbitrary point of the δ_1-neighborhood of t_0 and $\tau = t - t_0$. The norm of the expression in brackets does not exceed $\varepsilon/4$ because the points $t^{(m)} + \tau$ and $t^{(m)}$ lie in the δ-neighborhood of $\varkappa_0$:

$$\rho(t^{(m)}, \varkappa_0) < \delta/2, \quad \rho(t^{(m)} + \tau, \varkappa_0) \leqslant \rho(t^{(m)} + \tau, t^{(m)}) + \rho(t^{(m)}, \varkappa_0) = \rho(t + \tau, t) + \rho(t^{(m)}, \varkappa_0) < \delta/2 + \delta_1 \leqslant \delta.$$

We are going to estimate the norm of the second term:

$$\left\| \int_{t^{(m)}}^{t_0} [x(\eta + \tau) - x(\eta)]\, d\eta \right\| \leqslant (t_0 - t^{(m)}) \sup_\eta \|x(\eta + \tau) - x(\eta)\| \leqslant l \cdot \rho(\tau, 0) \leqslant l\delta_1 \leqslant \varepsilon/4.$$

Thus, $\|X(t) - X(t_0)\| < \varepsilon/2$, and so the oscillation of X(t) is less than ε. Since δ_1 does not depend on the choice of $t_0 \in J$, X(t) is uniformly continuous on J, and so on K.

LEMMA 2. If in the Banach space E there is a nonconvergent series $\sum x_k$, all the partial sums of which are bounded

$$\left\| \sum x_{k_i} \right\| \leqslant A < \infty,$$

then E contains a subspace isomorphic to c.

This result is due to A. Pelczynski [5] (see also [6]).

LEMMA 3. If a function F(s) with values in a Banach space is weakly continuous on some compact S, then on S there is at least one point where F(s) is strongly continuous.

This result is due to I. M. Gel'fand [7].

Proof of Theorem 1. Let us assume that the integral X(t) is not an almost periodic function but that $\|X(t)\| \le G < \infty$. By Lemma 1 the function X(t) is discontinuous at the zero of the set J. We form a sequence $\{t_n\}_1^\infty \subset J$ such that

$$\|X(t_n)\| \geqslant a > 0, \quad \rho(t_n, 0)\sum_{k=1}^{n-1}|t_k| < 2^{-n}. \tag{4}$$

Let $\tau_k = t_{n_k}$ (k = 1, 2, ..., m) be arbitrary terms of the sequence $\{t_n\}$. We consider the identity

$$X\left(\sum_1^m \tau_k\right) - \sum_1^m X(\tau_k) = \int_0^{\tau_1}[x(\eta+\tau_2)-x(\eta)]\,d\eta + \int_0^{\tau_1+\tau_2}[x(\eta+\tau_3)-x(\eta)]\,d\eta + \ldots + \int_0^{\tau_1+\ldots+\tau_{m-1}}[x(\eta+\tau_m)-x(\eta)]\,d\eta, \tag{5}$$

which, as well as the identity (3), follows directly from (1). By (4) and (5) we obtain that $\left\|\sum_1^m X(\tau_k)\right\| \leq G + 1$ $< \infty$. Thus, the vectors $X(t_n)$ (n = 1, 2, ...) form a divergent series whose partial sums are bounded. Hence, by Lemma 2, E contains a subspace isomorphic to c. Example (2) given above testifies to the validity of the second part of the theorem.

Proof of Theorem 2. Suppose that x(t) is an almost periodic function and that the set of values of X(t) is weakly compact (and hence is bounded). We take an arbitrary linear functional $f \in E^*$ and apply it to x(t) and to X(t):

$$\Phi(t) = \langle f, X(t)\rangle = \left\langle f, \int_0^t x(\eta)\,d\eta\right\rangle = \int_0^t \langle f; x(\eta)\rangle\,d\eta = \int_0^t \varphi(\eta)\,d\eta.$$

The function $\varphi(t) = \langle f, x(t)\rangle$ is almost periodic and

$$|\varphi(t') - \varphi(t'')| \leqslant \|f\| \cdot \|x(t') - x(t'')\|. \tag{6}$$

By the Bohl-Bohr theorem $\Phi(t)$ is an almost periodic function. It follows from (6) that it is uniformly continuous on J. In other words, the function X(t) is weakly uniformly continuous on J. Because the set of values of X(t) is weakly compact we can extend the definition of this function to K as the weakly continuous function $X(\varkappa) = \omega\text{-}\lim_{n\to\infty} X(t_n)$ $(\rho(t_n; \varkappa) \to 0)$, where "$\omega$-lim" denotes the weak limit. According to Lemma 3, $X(\varkappa)$ has at least one point of strong continuity and, consequently, by Lemma 1, it is strongly continuous on K. This in turn means that X(t) is almost periodic.

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