PROOF OF THE TOPOLOGICAL EQUIVALENCE OF ALL SEPARABLE INFINITE-DIMENSIONAL BANACH SPACES

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In 1928, Fréchet [1] raised the question: are all separable infinite-dimensional Banach spaces homeomorphic? In 1929, S. Mazur [2] proved that all spaces L_p and l_p $(1 \leq p < \infty)$ are homeomorphic. This was historically the first example of not isomorphic, but homeomorphic, Banach spaces. In 1933, S. Kaczmarz generalized Mazur's result to Orlich space [3]. Banach [4] repeated Fréchet's question, and concentrated on certain particular cases of this problem. From 1932 until 1953, only one paper [5] related to the Fréchet-Banach problem was published.

From 1953 to 1960, the author of the present paper published a number of notes [6-11], in which the homeomorphism of certain separable B-spaces was established. The result of [9] – the homeomorphism of all separable conjugate B-spaces – was simultaneously obtained by Klee [12]. The results of [6-9, 11 and 12] were established via two methods, which we can call: the method of equivalent norms and the method of coordinates. In 1960, Bessaga and Pelczynski [13] proved the following theorem:

If an infinite-dimensional separable B-space contains a subspace homeomorphic to l_2, or allows a linear continuous mapping onto a space homeomorphic to l_2, then X is homeomorphic to l_2.

The method used by Bessaga and Pelczynski is due to Borsuk [14] and may be called the method of expansion.

The present paper is devoted to the detailed proof of a theorem which supplies a positive answer to the Fréchet-Banach problem:

<u>THEOREM</u>. All separable infinite-dimensional Banach spaces are topologically equivalent.

For the proof of the theorem, all the methods of proof just cited will be utilized.

In shortened form, this proof was published in [15], and was presented at the International Congress of Mathematicians at Moscow.

We shall not touch here on the broader problems of topological classification of F-spaces and their subsets. The basic results and the open questions relating to these problems were included in a synoptic paper of Bessaga [16] (cf., also, the abstracts of the papers of Bessaga, Pelczynski, and Klee, and the paper of Anderson at the Congress).

§ 1. EQUIVALENT NORMS

Let X be a real B-space with basis $\{e_k\}_1^\infty$; we denote by $\{f_k\}_1^\infty$ a system of linear functionals conjugate to the basis. Thus, each element $x \in X$ is represented in the form

$$x = \sum_{k=1}^{\infty} f_k(x) e_k.$$

We introduce the further notation:

$$S_n(x) = \sum_{k=1}^{n} f_k(x) e_k; \quad R_n x = \sum_{k=n+1}^{\infty} f_k(x) e_k.$$

While denoting the original norm of space X by $\|\cdot\|_0$, we introduce the equivalent norm:

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$$\| x \|_1 = \sup_{m,n} \left\| \sum_{m+1}^{n} f_k(x) e_k \right\|_0 .$$

As is easily verified, the norm $\| \cdot \|_1$ possesses the following monotonicity property: for all m, n ($1 \leq m < n \leq \infty$) and for all coefficients λ_k,

$$\left\| \sum_{m+1}^{n} \lambda_k e_k \right\|_1 \leqslant \left\| \sum_{m}^{n+1} \lambda_k e_k \right\| . \tag{1}$$

We introduce the following equivalent norm by:

$$\| x \|_2 = \sum_{n=0}^{\infty} 2^{-n} \| R_n x \|_1 . \tag{2}$$

This norm retains property (1). We now prove that, in addition, it satisfies the conditions: if for some x_ν and x

$$\lim_{\nu \to \infty} f_n(x_\nu) = f_n(x) \quad (n = 1, 2, \ldots), \tag{3}$$

then

$$\varliminf_{\nu \to \infty} \| x_\nu \|_2 \geqslant \| x \|_2 ; \tag{4}$$

and if, in addition to (3), there holds the following condition

$$\lim_{\nu \to \infty} \| x_\nu \|_2 = \| x \|_2 , \tag{5}$$

then

$$\lim_{\nu \to \infty} \| x_\nu - x \|_2 = 0. \tag{6}$$

In other words, the unit solid sphere U $\{ \| x \|_2 \leq 1 \}$ is closed with respect to coordinatewise convergence, while on the unit spherical surfaces S $\{ \| x \|_2 = 1 \}$, coordinatewise convergence coincides with strong convergence.

LEMMA 1. It follows from conditions (3) that

$$\varliminf_{\nu \to \infty} \| R_n x_\nu \|_1 \geqslant \| R_n x \|_1 \qquad (n = 0, 1, 2, \ldots). \tag{7}$$

Proof. We fix n, and take some arbitrary $\varepsilon > 0$. We choose m so large that

$$\| R_n x - S_m R_n x \|_1 < \frac{\varepsilon}{2} . \tag{8}$$

Further, we choose ν_0 such that, for all $\nu \geq \nu_0$,

$$\| S_m R_n x_\nu - S_m R_n x \|_1 < \frac{\varepsilon}{2} . \tag{9}$$

We thus find from (8) and (9) that

$$\| R_n x - S_m R_n x_\nu \|_1 < \varepsilon$$

which means

$$\| R_n x \|_1 - \varepsilon < \| S_m R_n x_\nu \|_1 \leqslant \| R_n x_\nu \|_1,$$

and this proves the lemma.

Property (4) follows from (2) and (7). Confronting (2), (5) and (7), we see that conditions (3) and (5) entail the following system of equations:

$$\lim_{\nu\to\infty} \| R_n x_\nu \|_1 = \| R_n x \|_1 \qquad (n = 0, 1, 2, \ldots). \tag{10}$$

LEMMA 2. The set $\{x_\nu\}$, subject to condition (10), is compact.

Proof. Assigning some $\varepsilon > 0$, we define n_0, and then ν_0, such that

$$\| R_{n_0} x \|_1 < \frac{\varepsilon}{2}; \ \| R_{n_0} x_\nu - R_{n_0} x \| < \frac{\varepsilon}{2} \qquad (\nu \geqslant \nu_0),$$

whence

$$\| R_{n_0} x_\nu \|_1 < \varepsilon \qquad (\nu \geqslant \nu_0).$$

We replace n_0 by a larger subscript, n_1, such that the last inequality is extended to $\nu < \nu_0$. In accordance with the monotonocity condition (1) for the basis, we obtain

$$\| R_n x_\nu \|_1 < \varepsilon \ (\nu = 1, 2, \ldots; \ n \geqslant n_1(\varepsilon)),$$

i.e., the norm of the remainder of the basis expansion tends to zero as $n \to \infty$, uniformly in ν. The proof is completed by citing the criterion for compactness in B-spaces with bases [17, page 247].

Compactness of the sequence $\{x_\nu\}$, in conjuction with coordinatewise convergence, entails strong convergence. Thus, the properties of norm $\| \cdot \|_2$ are verified. This norm was introduced in [18].

Finally, we construct the equivalent norm $\| \cdot \|$ which will occur in what follows:

$$\| x \| = \sqrt{\| x \|_2^2 + J^2(x)}; \qquad J(x) = \sqrt{\sum_1^\infty 2^{-k} \left(\frac{f_k(x)}{\| f_k \|_2} \right)^2}. \tag{11}$$

It is obvious that this norm also has property (1). We now verify that, for it, (6) again follows from (3) and (5). Indeed, let (3) hold and

$$\lim_{\nu\to\infty} \| x_\nu \| = \| x \|. \tag{5a}$$

It follows from the definition of J(x) that

$$\varliminf_{\nu\to\infty} J(x_\nu) \geqslant J(x). \tag{12}$$

Since, in addition, (4) follows from (3) then, confronting (4), (12), (11) and (5a), we see that

$$\lim_{\nu\to\infty} \| x_\nu \|_2 = \| x \|_2 \tag{5}$$

and this means that, from the properties of norm $\| \cdot \|_2$, (6) holds. Since the norms $\| \cdot \|$ and $\| \cdot \|_2$ are equivalent,

$$\lim_{\nu\to\infty} \| x_\nu - x \| = 0.$$

Definition. A Banach space is called locally uniformly convex if, from the conditions

$$\|x_\nu\| = \|x\| = 1, \quad \lim_{\nu\to\infty} \|x_\nu + x\| = 2 \tag{13}$$

it follows that

$$\lim_{\nu\to\infty} \|x_\nu - x\| = 0.$$

We now prove that space X is locally uniformly convex with respect to the norm of (11). In accordance with (11), we rewrite conditions (13) in the form

$$\|x_\nu\|_2^2 + J^2(x_\nu) = \|x\|_2^2 + J^2(x) = 1, \tag{14}$$

$$\lim_{\nu\to\infty} [\|x_\nu + x\|_2^2 + J^2(x_\nu + x)] = 4. \tag{15}$$

We add the two evident relationships:

$$J^2(x_\nu - x) + J^2(x_\nu + x) = 2[J^2(x_\nu) + J^2(x)],$$

$$\|x_\nu + x\|_2^2 \leqslant 2[\|x\|_2^2 + \|x_\nu\|_2^2]$$

and obtain

$$J^2(x_\nu - x) + [\|x_\nu + x\|_2^2 + J^2(x_\nu + x)] \leqslant 2[\|x_\nu\|_2^2 + J^2(x_\nu)] + 2[\|x\|_2^2 + J^2(x)]. \tag{16}$$

Confronting (14), (15) and (16), we obtain

$$\lim_{\nu\to\infty} J(x_\nu - x) = 0. \tag{17}$$

It follows from (17) that

$$\lim_{\nu\to\infty} J(x_\nu) = J(x); \ \lim_{\nu\to\infty} f_n(x_\nu) = f_n(x) \qquad (n = 1, 2, \ldots). \tag{18}$$

By scrutinizing (14) and the first of conditions (18), we see that

$$\lim_{\nu\to\infty} \|x_\nu\| = \|x\|_2. \tag{5}$$

For $\|\cdot\|_2$, from coordinatewise convergence and convergence of the norm follow strong convergence, which also proves the locally uniform convexity of space (X, $\|\cdot\|$). The norm of (11) was considered in [19]. Proof of the equivalence of all the norms considered here presents no difficulty.

Summarizing all we have proven in this section, we have

ASSERTION 1. In a Banach space with basis $\{e_k\}_1^\infty$, there exists an equivalent norm, $\|\cdot\|$, possessing the following properties:

a) the basis with respect to this norm is orthogonal:

$$\left\|\sum_{k=1}^{n-1} a_k e_k\right\| < \left\|\sum_{k=1}^{n} a_k e_k\right\| \qquad (a_n \neq 0;\ n = 1, 2, \ldots);$$

b) on the unit spherical surface, coordinatewise convergence coincides with convergence in norm;

c) the space is locally uniformly convex.

§2. LOCAL MODULES OF CONVEXITY

Consider the functional

$$\omega(x, \delta) = \frac{1}{2} \sup_{z \in G(x,\delta)} \|x - z\| \qquad (\|x\| = 1;\ 0 \leqslant \delta \leqslant 1), \tag{19}$$

where

$$G(x, \delta) = \left\{ z:\quad \|z\| \leqslant 1;\quad \min_{0 \leqslant \lambda \leqslant 1} \|\lambda x + (1 - \lambda) z\| \geqslant 1 - \delta \right\} \tag{20}$$

For all δ, the local module of convexity, $\omega(x, \delta)$, satisfies the inequalities

$$\delta \leqslant \omega(x, \delta) \leqslant \omega(x, \delta_1) \leqslant \omega(x, 1) = 1 \qquad (\delta \leqslant \delta_1 \leqslant 1).$$

If the space is locally uniformly convex, then

$$\lim_{\delta \to 0} \omega(x, \delta) = 0. \tag{21}$$

LEMMA 3. A local module of convexity satisfies the conditions

$$\omega(x, \delta + h) - \omega(x, \delta) \leqslant \frac{2h}{\delta^2} \qquad (0 < \delta \leqslant \delta + h \leqslant 1), \tag{22}$$

$$\omega(x, \delta) \leqslant \frac{1}{2} \|x - y\| + \omega(y, \delta + \|x - y\|) \quad (0 < \delta < \delta + \|x - y\| \leqslant 1). \tag{23}$$

Proof. In accordance with the definition of a local module of convexity, it suffices to verify inegual-ity (22) for an arbitrary two-dimensional section of the unit sphere containing the center of the sphere and the point x. Reduction to a two-dimensional space allows us to have recourse to illustrations. Figure 1 represents concentric spheres of radii 1 and 1-δ. The set G(x, δ) is cross-hatched. The distance from the line xz_1 to the center, θ, equals 1-δ-h. It is necessary to find an upper bound for the difference $\|x - z_1\| - \|x - z\|$. We introduce the coordinate system: the $\theta\xi$ axis passes through point v of the tangent chord xz and the inner circle; the $\theta\eta$ axis is parallel to chord xz. We give now the coordinates of the points we shall need:

$$x(1 - \delta, \omega^+);\ y(1 - \delta, -\omega^-);$$
$$u(1 - \delta_1, 0);\ v(1 - \delta, 0);\ w(1, 0),$$

where ω^+ and ω^- are the lengths of segments xv and vz such that

$$\delta \leqslant \omega^{\pm} \leqslant 1;\ \delta \leqslant \delta_1 \leqslant \delta + h; \qquad \omega^+ + \omega^- = 2\omega.$$

Figure 2 shows the unit sphere which is "worst" for the given $\omega^{\pm}$ and δ : on it, $\|x - z_1\|$ attains the greatest of its possible values. Using the ordinary tools of affine analytic geometry (we omit the corre-sponding horrendous computations), we can obtain

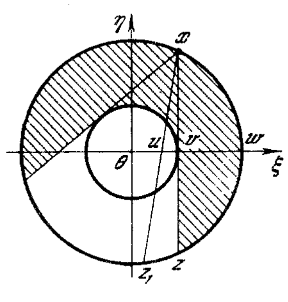

Fig. 1

$$\|x - z_1\| = 2 \min \left\{ 1;\ \omega \cdot \frac{\delta}{1 - \delta} \cdot \frac{(1 - \delta - h)\omega^+ + h}{\delta\omega^+ - h\omega^-} \right\},$$

whence, after a number of transformations designed to eliminate the quantities ω^+ and ω^- and to simplify the resulting expressions, we arrive at inequality (22).

It is more convenient to establish inegulality (23) analytically. Let x and y be close points on the unit sphere ($\|x - y\| < 1 - \delta$). From the inequality

$$\|\lambda y + (1 - \lambda) z\| \geqslant \|\lambda x + (1 - \lambda) z\| - \lambda \|x - y\| \geqslant 1 - (\delta + \|x - y\|)\ (0 \leqslant \lambda \leqslant 1)$$

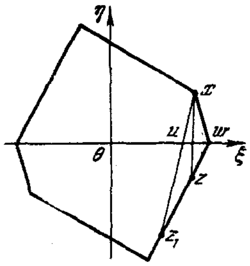

Fig. 2

follows the set inclusion

$$G(x,\delta)\subset G(y,\delta+\|x-y\|),$$

from whence, in accordance with the definition of $\omega(x, \delta)$, we obtain inequality (23).

From (22), (23) and (21) follows immediately the following, which we shall need in the sequel:

ASSERTION 2. In any Banach space, a local module of convexity $\omega(x, \delta)$ is uniformly continuous on the set $S \times [\delta_0 : 1]$ (S is the unit sphere of the space). If the space is locally uniformly convex, then $\omega(x, \delta)$ is, moreover, continuous on the set $S \times [0; 1]$ $(0<\delta_0<1)$.

The results of this section were obtained by the author in collaboration with V. I. Gurarie.

§3. AUXILIARY CONSTRUCTIONS

On the unit solid sphere of space X we construct the functional

$$\Phi(x)=\omega\left(\frac{x}{\|x\|};\ 1-\|x\|\right);\ \Phi(\theta)=1. \tag{24}$$

It follows from the results of §2 that this functional is continuous on the unit sphere $U\{\|x\|\le 1\}$ and is uniformly continuous on each sphere $U\{\|x\|\le 1-\delta_0\}$; on the unit sphere, the functional $\Phi(x) = 0$, and inside it, the following inequality is valid

$$1-\|x\|\leqslant\Phi(x)\leqslant 1.$$

LEMMA 4. If the numbers $\{a_k\}_1^\infty$ are such that

$$\lim_{n\to\infty}\Phi(s_n)=0 \qquad \left(s_n=\sum_1^n a_k e_k\right), \tag{25}$$

then the series $\Sigma a_k e_k$ converges.

Proof. It follows from conditions (1) and (25) that

$$\|s_1\|\leqslant\|s_2\|\leqslant\ldots;\quad \lim_{n\to\infty}\|s_n\|=1. \tag{26}$$

we now show that, for any n, all elements s_m $(m\ge n)$ lie in the set $G(s_n/\|s_n\|;\ 1-\|s_n\|)$. Indeed,

$$\|\lambda s_n\cdot\|s_n\|^{-1}+(1-\lambda)s_m\|\geqslant\|\lambda s_n\cdot\|s_n\|^{-1}+(1-\lambda)s_n\|=\|s_n\|\cdot[\lambda\|s_n\|^{-1}+(1-\lambda)]\geqslant\|s_n\| \quad (0\leqslant\lambda\leqslant 1),$$

whence the requisite inclusion also follows. Condition (25) means that the diameter of the set $G(s_n/\|s_n\|;\ 1-\|s_n\|)$ tends to zero as $n\to\infty$. This means that the sequence $\{s_n\}_1^\infty$ is fundamental and, thus, the series $\Sigma a_k e_k$ converges to some normal element.

For each element $x\in U\{\|x\|\le 1\}$, we consider the broken line (generally speaking, infinitely segmented), successively linking the points θ, S_1x, S_2x, ...; we add to this the element x itself, and we denote by L(x) the closed set thus obtained (homeomorphic to a segment). We define the functional F(x), which will participate in subsequent constructions:

$$F(x)=\left(1-\frac{1}{2}\|x\|\right)\min_{z\in L(x)}\Phi(z) \qquad (\|x\|\leqslant 1). \tag{27}$$

This functional is continuous, satisfies the inequality

$$\left(1-\frac{1}{2}\|x\|\right)(1-\|x\|)\leqslant F(x)\leqslant 1 ,$$

and vanishes on the unit sphere.

We shall now prove that Lemma 4 also holds for functional F(x). Let

$$\lim_{n\to\infty} F(s_n)=0 \qquad \left(s_n=\sum_1^n a_k e_k\right).$$

This means that, for each n, we can find $\nu = \nu(n) \le n$ such that

$$\lim_{n\to\infty} \Phi(s_{\nu-1}+\lambda_\nu e_\nu)=0 \qquad (0\leqslant|\lambda_\nu|\leqslant|a_\nu|).$$

According to (26), ν increases without bound as n increases. Repeating almost word for word the discussion of Lemma 4, we find that the expression $s_{\nu-1}+\lambda_\nu e_\nu$ tends with increasing ν to some normed element, the basis expansion of which is the series $\Sigma a_k e_k$.

ASSERTION 3. On the unit sphere of space X it is possible to define a continuous functional, F(x), possessing the following properties:

a) $F(x) > 0$ for $\|x\| < 1$; $F(x) = 0$ for $\|x\| = 1$; $F(\theta) = 1$;

b) if $\lim_{n\to\infty} F\left(\sum_1^n a_k e_k\right)=0$, then the series $\Sigma a_k e_k$ converges;

c) for fixed n and $\{a_k\}_1^{n-1}$, the function

$$\psi(\alpha)=F\left(\sum_1^{n-1} a_k e_k+\alpha e_n\right)$$

is strictly increasing for $\alpha < 0$ and is strictly decreasing for $\alpha > 0$.

Proof. F(x) defined by formula (27) is such a functional. Properties (a) and (b) are already established. Let us prove (c). Let $|\alpha_1|<|\alpha_2|$; $\alpha_2\alpha_2 \ge 0$. Then,

$$\|s_{n-1}+\alpha_1 e_n\|<\|s_{n-1}+\alpha_2 e_n\| \tag{28}$$

by virtue of the orthogonality of the basis, and

$$L(s_{n-1}+\alpha_1 e_n)\subset L(s_{n-1}+\alpha_2 e_n), \tag{29}$$

by definition of the set L(x). From (27)-(29) we get $\psi(\alpha_1) > \psi(\alpha_2)$.

§ 4. HOMEOMORPHISM OF SPACES WITH BASES

To each normed element $x \in X$ we put into correspondence the numerical sequence

$$h_n(x)=[F^2(S_{n-1}x)-F^2(S_n x)]^{1/2}\operatorname{sign} f_n(x) \qquad (n=1, 2, \ldots). \tag{30}$$

LEMMA 5. If x is an element of the unit sphere, then $\sum_1^\infty h_n^2(x)=1$. For any real numbers $\{h_n\}_1^\infty$ subject to the condition that $\Sigma h_n^2 = 1$, we can find a unique normed element x such that

$$h_n(x)=h_n \qquad (n=1, 2, \ldots).$$

Proof. The first part of the lemma is directly verified. We turn to its second part. We choose the coefficient a_1 such that

$$1 - F^2(a_1 e_1) = h_1^2, \qquad \operatorname{sign} a_1 = \operatorname{sign} h_1. \tag{31_1}$$

When the coefficients $\{a_k\}_1^{n-1}$ are all defined, we then define a_n from the conditions

$$F^2(s_{n-1}) - F^2(s_{n-1} + a_n e_n) = h_n^2, \qquad \operatorname{sign} a_n = \operatorname{sign} h_n. \tag{31_n}$$

According to property (c) of functional F(x), each coefficient, a_k, is determined uniquely. Adding (31_n) and taking into account the condition that $\Sigma h_n^2 = 1$, we convince ourselves that $\lim_{n\to\infty} F(s_n) = 0$. This means that, by property (b), the series $\Sigma a_k e_k$ converges, and its sum is the normed element x being sought.

LEMMA 6. The normed sequence x_ν converges to the element x if and only if

$$\lim_{\nu\to\infty} h_n(x_\nu) = h_n(x) \qquad (n = 1, 2, \ldots). \tag{32}$$

Proof. If $x_\nu \to x$, then (32) is a consequence of the continuity of F(x). Now, let (32) hold. Considering this equation successively for n = 1, 2, ..., we convince ourselves that $\lim_{\nu\to\infty} f_n(x_\nu) = f_n(x)$ (n = 1, 2, ...). Since, moreover, $\|x_\nu\| = \|x\| = 1$ then, according to property (b), the norms of space X, $x_\nu \to x$.

ASSERTION 4. Space X is homeomorphic to space l_2.

Proof. It follows from Lemma 5 that, by putting into correspondence with each normed element $x \in X$ the sequence of its coordinates

$$Hx = \{h_n(x)\}_{n=1}^{\infty} \qquad (\|x\| = 1),$$

we arrive at a one-to-one correspondence between the spheres of spaces X and l_2. We now prove that this correspondence, H, is a homeomorphism. We note, for this, that the natural norm of space l_2 satisfies conditions a)–c) of Assertion 1, and that in l_2 one can set $F(y) = \sqrt{1 - \|y\|^2}$. We now consider the convergent sequence of normed elements of space X:

$$\lim_{\nu\to\infty} x_\nu = x; \qquad \|x_\nu\| = \|x\| = 1. \tag{33}$$

It follows from (33) that $\lim_{\nu\to\infty} h_n(x_\nu) = h_n(x)$ (n = 1, 2 . . .), whence, according to correspondence H,

$$\lim_{\nu\to\infty} h_n(y_\nu) = h_n(y) \quad (y_\nu = Hx_\nu;\ y = Hx). \tag{34}$$

According to Lemma 6, it follows from (34) that $\lim_{\nu\to\infty} y_\nu = y$, which proves the continuity of mapping H. Continuity of the inverse mapping, H^{-1}, is proven analogously.

Homeomorphism H is extended from spheres to the entire space by the formula

$$y = \|x\| \cdot H(x / \|x\|);\ H(\theta) = \theta \qquad (x \in X;\ y \in l_2).$$

Since equivalent changes of norms of a B-space do not affect its topology, it then follows from Assertion 4 that all infinite-dimensional B-spaces with bases are topologically equivalent.

§5. HOMEOMORPHISM OF ALL SEPARABLE INFINITE-DIMENSIONAL B-SPACES

It is now necessary to extend the result of Assertion 4 to spaces without bases (so far, their existence has been neither proven nor even verified).

Since each infinite-dimensional B-space contains an infinite-dimensional subspace with a basis, the target homeomorphism follows from the Bessaga-Pelczynski theorem formulated in our introduction.

For completeness of exposition, we shall prove the result we need here.

We introduce the product of B-spaces with a countable or finite number of factors:

$$Z = Z_1 \times Z_2 \times Z_3 \times \ldots$$

This is a B-space, the elements of which are the sequences

$$z = \{z_1, z_2, z_3, \ldots\}, \quad z_n \in Z_n, \quad \lim_{n\to\infty} \| z_n \| = 0,$$

with the norm $\|z\| = \max_n \|z_n\|$, and with term-by-term addition and multiplication by a scalar. We note the following isometric (and, moreover, homeomorphic) correspondence:

$$c_0 \equiv c_0 \times c_0 = c_0 \times c_0 \times c_0 \times \ldots, \tag{35}$$

where c_0 is the B-space of all numerical sequences which converge to zero.

We now state without proof the assertion of [12], which is a simple corollary of the Bartle-Graves theorem [20].

LEMMA 7. If Z is a Banach space and Z_1 a subspace of it, then

$$Z \sim Z_1 \times Z/Z_1$$

(where the symbol ~ denotes homeomorphism).

Finally, we consider an infinite-dimensional separable B-space, X, with no additional constraints imposed. Let Y be its infinite-dimensional subspace with a basis; we denote by Z the factor-space X/Y. Since spaces c_0 and C (the spaces of functions continuous on segments) have bases, then

$$Y \sim C \sim c_0 \sim c_0 \times c_0 \sim c_0 \times c_0 \times c_0 \times \ldots \tag{36}$$

By using Lemma 7 and relationship (36), we obtain

$$X \sim Y \times Z \sim (c_0 \times c_0) \times Z \sim c_0 \times (c_0 \times Z) \sim c_0 \times (Y \times Z) \sim C \times X. \tag{37}$$

Since C is a universal space ([17], page 256), it contains a subspace, X_1, isometric to X. Therefore,

$$C \sim X \times W \sim c_0 \qquad (W = C / X_1). \tag{38}$$

From (36) and (38), we get

$$\begin{aligned} C &\sim c_0 \times c_0 \times c_0 \times \ldots \sim (X \times W) \times (X \times W) \times (X \times W) \times \ldots \\ &\sim X \times (W \times X) \times (W \times X) \times \ldots \sim X \times (c_0 \times c_0 \times c_0 \times \ldots) \sim X \times C. \end{aligned} \tag{39}$$

Taking both (37) and (39) into account, we arrive at the required homeomorphism:

$$X \sim C.$$

We have thus proven that all separable infinite-dimensional Banach spaces (with bases or without them) are homeomorphic.

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of the first issue of this year.
