NORMING SUBSPACES, BIORTHOGONAL SYSTEMS, AND PRECONJUGATE BANACH SPACES

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The subspace Γ of the conjugate space X^* of the Banach space X is called norming, if its Dicksme characteristic

$$r(\Gamma) = \inf\left\{\left\{\sup\frac{|f(x)|}{\|f\|\cdot\|x\|} : f \in \Gamma\right\} : x \in X\right\}$$

is strictly greater than zero. This characteristic of the subspace Γ may also be calculated by the following formula [1]:

$$r(\Gamma) = \inf\left\{\frac{\|F - x\|}{\|x\|} : x \in X,\ F \in \Gamma^{\perp}\right\} = \frac{1}{\|P_\Gamma\|},$$

where P_Γ is a projector from $\pi(X) \oplus \Gamma^{\perp}$ onto $\pi(X)$, parallel to $\Gamma^{\perp}$, the annihilator of Γ in the space X^{**} (π is the natural embedding of X in X^{**}). For subsets $U \subset X$, $V \subset X^*$, we define the characteristic of V relative to U:

$$r(V,V) = \inf\left\{\sup\left\{\frac{|v(u)|}{\|v\|\cdot\|u\|} : v \in V\right\} : u \in U\right\}.$$

In the case when U and V are subspaces of X and X^*, respectively, then $r(V, U) = \|P\|^{-1}$, where P is a projector from $U \oplus V_\perp$ onto U, parallel to $V_\perp$, the annihilator of V in the space X [2].

As van Dulst and Singer showed [3], if Γ is a separable subspace of X^*, then on X there exists an equivalent norm such that in this norm the characteristic of each subspace not containing Γ is strictly less than one. We will show that there exists an equivalent norm on X with this property for nonseparable subspace $\Gamma \subset X^*$.

We recall that the system of elements $\{x_i\}_{i\in I}$ of the Banach space X is called minimal, if there exists a system of linear functionals $\{x_i^*\}_{i\in I} \subset X^*$ (called the conjugate of the original system), such that $x_i^*(x_j) = \delta_{i,j}$. A minimal system together with its conjugate, $\{x_i, x^{i*}\}_{i\ni I}$, is called a biorthogonal system. A biorthogonal system is called bounded, if $\sup\|x_i\|\cdot\|x_i^*\| < \infty$.

THEOREM 1. Let $\{x_i, x_i^*\}_{i\in I}$ be a bounded biorthogonal system in the Banach space X, and let $\Gamma = [x_i^*]$. Then on X there exists an equivalent norm such that in the new norm, for any subspace $G \subset X^*$ not containing Γ, $r(G, \{x_i\}_{i\in I}) < 1$.

Proof. We shall assume that $\|x_i\|_0 = 1$ $(i \in I)$ and $\sup\|x_i^*\|_0 \leqslant C < \infty$, where $\|\cdot\|_0$ is the original norm on X. We introduce a system of projectors, connected with the system $\{x_i, x_i^*\}_{i\in I}$:

$$P_i(x) = x_i^*(x)\,x_i,\quad Q_i(x) = x - P_i(x)\quad (x \in X).$$

Clearly, $\|P_i\|_0 = \|P_i^*\|_0 = \|x_i\|\cdot\|x_i^*\| \le C$. On X we introduce a new norm $\|\cdot\|$, setting for each $x \in X$

$$\|x\| = \max\left\{\frac{C}{C+1}\|x\|_0,\ \sup_{i\in I}\|P_i(x)\|_0\right\}.$$

Clearly, the norm $\|\cdot\|$ is equivalent to the original norm:

$$\frac{C}{C+1}\|x\|_0 \leqslant \|x\| \leqslant C\cdot\|x\|_0.$$

LEMMA 1. For any $i \in I, \|x_i\| = \|x_i^*\| = \|P_i\| = \|P_i^*\| = 1$.

Proof. Since $\|x_i\|_0 = 1$, then $\|x_i\| = \max\{c/(c+1), 1\} = 1$. We now calculate $\|x_i^*\|$:

$$x_i^*\| \leqslant \sup\left\{\frac{|(x_i^*, x)|}{\|x\|} : x = x_i + y,\ P_i(y) = 0\right\} = \sup\left\{\frac{|(x_i^*, x_i + y)|}{\|x_i + y\|} : P_i(y) = 0\right\} =$$

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$$= \sup\left\{\frac{1+|(x_i^*, y)|}{\|x_i+y\|}: P_i(y)=0\right\} = \sup\left\{\frac{1}{\|x_i+y\|}: P_i(y)=0\right\}.$$

But if $P_i(y) = 0$, then $\|x_i + y\| \geq \|P_i(x_i + y)\|_0 = \|x_i\|_0 = 1$. Thus $\|x_i^*\| \leq 1$. On the other hand, $\|x_i^*\| \geq |(x_i^*, x_i)|/\|x_i\| = 1$. Thus $\|x_i\| = \|x_i^*\| = 1$ and hence $\|P_i\| = \|P_i^*\| = \|x_i\| \cdot \|x_i^*\| = 1$.

LEMMA 2. If for some $i \in I$ and $x \in X$ $P_i(x) = x_i$ and $\|Q_i(x)\| \leq 1/(C+1)$, then $\|x\| = 1$.

Proof. Since $P_i(x) = x_i$, the element x may be written in the form $x = x_i + Q_i(x)$. Then

$$\|x\| = \max\left\{\frac{C}{C+1}\|x_i + Q_i(x)\|_0, \sup_j \|P_j(x_i) + P_j \cdot Q_i(x)\|_0\right\} \leqslant \max\left\{\frac{C}{C+1}(1+\|Q_i(x)\|_0), \|P_i(x)\|_0, \sup_{j\neq i}\|P_jQ_i(x)\|_0\right\}$$

$$\leqslant \max\left\{\frac{C}{C+1}\left(1+\frac{C+1}{C}\|Q_i(x)\|\right), 1, C\cdot\frac{C+1}{C}\|Q_i(x)\|\right\} = \max\left\{\frac{C}{C+1}+\|Q_i(x)\|, 1, (C+1)\|Q_i(x)\|\right\} \leqslant 1.$$

On the other hand, $\|x\| \geq \|P_i(x)\|_0 = \|x_i\|_0 = 1$.

LEMMA 3. For any $i \in I$ and each $x^* \in X$

$$\|P_i^*(x^*)\| + \frac{1}{2(C+1)}\|Q_i^*(x^*)\| \leqslant \|x^*\|.$$

Proof. Fix $i \in I$; it is sufficient to establish the required inequality only for those $x^* \in X^*$ for which $|x^*(x_i)| > 0$. Using Lemmas 1 and 2, we obtain $\|x^*\| = \sup\{|(x^*, x)| : \|x\| \leqslant 1\} \geqslant \sup\{|(x^*, x)| : P_i(x) = x_i, \|Q_i(x)\| \leqslant 1/(C+1)\} \geqslant |(x^*, x_i)| + \sup\{|(x^*, Q_i(x)| : \|Q_i(x)\| \leqslant 1/(C+1)\} \geqslant \|P_i^*(x^*)\| + \sup\{|Q_i^*(x^*), x)| : \|x\| \leqslant 1/2(C+1)\} \geqslant \|P_i^*(x^*)\| + \frac{1}{2(C+1)}\|Q_i^*(x^*)\|$.

LEMMA 4. In the norm $\|\cdot\|$, each element x_i^* is a strictly exposed point of the unit sphere of the space X^*, i.e., if

$$\{f_n\}_{n=1}^\infty \subset X^*, \ \|f_n\| = 1 \ (n = 1, 2 \ldots), \ \lim_{n\to\infty} f_n(x_i) = 1, \tag{*}$$

then $\lim_{n\to\infty}\|f_n - x_i^*\| = 0$.

Proof. Let the sequence $\{f_n\}_{n=1}^\infty$ satisfy condition (*). Then, applying Lemma 3, we obtain

$$\|f_n - x_i^*\| \leqslant \|P_i^*(f_n) - x_i^*\| + \|Q_i^*(f_n)\| = \|f_n(x_i)x_i^* - x_i^*\|$$

$$+ \|Q_i^*(f_n)\| \leqslant |f_n(x_i) - 1| \cdot \|x_i^*\| + 2(C+1)(\|f_n\| - \|P_i^*(f_n)\|) = |f_n(x_i) - 1| + 2(C+1)(1 - |f_n(x_i)|) \to 0$$

as $n \to \infty$.

We return to the proof of the theorem. Let G be a subspace of X^* such that $r(G, \{x_i\}_{i\in I}) = 1$. For arbitrary $i \in I$, there exists a sequence $\{f_n\}_{n=1}^\infty \subset G$, $\|f_n\| = 1$, such that $\lim_{n\to\infty} f_n(x_i) = \|x_i\| = 1$. By Lemma 4 $\lim_{n\to\infty}\|f_n - x_i^*\| = 0$, i.e., $x_i^* \in G$. Thus the space G contains all the elements x_i^* $(i \in I)$, and hence contains the subspace Γ generated by them.

From this theorem, we immediately obtain:

COROLLARY 1. Let $\{x_i, x_i^*\}_{i\in I}$ be a bounded biorthogonal system in the Banach space X, and let $\Gamma = [x_i^*]$. Then on X there exists an equivalent norm such that in this new norm, for any subspace $G \subset X^*$ not containing Γ, $r(G) < 1$.

COROLLARY 2. Let $\{x_i, x_i^*\}_{i\in I}$ be a bounded biorthogonal spanning system in the Banach space X (i.e., $[x_i^*] = X^*$). Then on X there exists an equivalent norm such that in this new norm, $r(G) < 1$ for each subspace $G \subset X^*$.

Let $X = c_0(S)$; then $X^* = l_1(S)$ and $X^{**} = m(S)$. Denote by π_0 the natural imbedding of X in X^{**}, and by π_1 the natural imbedding of X^* in X^{***}.

COROLLARY 3. Let $Y = m(S)$ and G be a subspace of Y^* such that $r(G, \pi_0(c_0(S)) = 1$. Then G contains $\pi_1(l_1(S))$.

In fact, if $\{x_s\}_{s\in S}$ is a natural basis of the space $c_0(S)$, then the conjugate system $\{x_s^*\}_{s\in S}$ is a natural basis of the space $l_1(S)$. Thus $\{\pi_0(x_s), \pi_1(x_s^*)\}_{s\in S}$ is a bounded biorthogonal system in Y, and $[\pi_1(x_s^*)] = \pi_1(l_1(S))$.

Moreover, the equivalent norm on Y, constructed in Theorem 1, coincides to within a numerical multiple with the original norm on Y: $\|x\|_0 = \sup\{|x(s)| : s \in S\} = \sup\{|(\pi_1(x_s^*), x)| : s \in S\}$, $(x \in Y)$.

COROLLARY 4. Let Y be an arbitrary subspace of the separable Banach space X. Then on X there exists an equivalent norm $\|\cdot\|$, such that for each $x \in X \setminus Y$

$$\sup\left\{\frac{\|y\|}{\|\lambda x + y\|} : y \in Y,\ \lambda \in R\right\} > 1$$

(i.e., the norm of the projector $P: [x] \oplus Y \to Y$, parallel to [x], is strictly greater than one).

Proof. It is known [4] that the space Y has a complete minimal bounded system $\{y_i\}_{i=1}^{\infty}$. By a result of Singer [5], there exists a conjugate $\{y_i\}_{i=1}^{\infty}$ to the system $\{y_i^*\}_{i=1}^{\infty} \subset X^*$ such that $\sup \|y_i\| \cdot \|y_i^*\| < \infty$ and $[y_i^*]_{\perp} \subset Y$. We introduce an equivalent norm on X in the same way as in Theorem 1 (for the system $\{y_i, y_i^*\}_{i=1}^{\infty}$). For some $x \notin Y$, let

$$\sup\left\{\frac{\|y\|}{\|\lambda x + y\|} : y \in Y,\ \lambda \in R\right\} = \|P\| = 1.$$

Write $G = [x]^{\perp}$. Then by Theorem 1, $[y_i^*] \subset G$ and therefore $G_{\perp} \subset [y_i^*]_{\perp} = Y$. But $G_{\perp} = ([x]_{\perp})^{\perp} = [x]$, i.e., $x \in Y$, which contradicts our assumption.

The Banach space X is said to be the unique preconjugate to within isometry to its conjugate X*, if each Banach space Y such that Y* is isometric to X*, is isometric to X. In this case we also say that the space X* has unique preconjugate to within isometry. Grotendieck's well-known theorem [6] states that conjugate spaces of continuous functions on a bicompactum have unique preconjugate to within isometry. Some subspaces and factor-spaces of functional spaces are the unique preconjugate to their conjugate spaces [7, 8]. On the other hand, the space l_1 has an uncountable family of pairwise nonisomorphic preconjugate spaces [9].

THEOREM 2. Let the Banach space X have a complete biorthogonal bounded system $\{x_i, x_i^*\}_{i \in I}$. Then on X there exists an equivalent norm such that in this new norm the space X is the unique preconjugate to within isometry to its conjugate.

Proof. Without loss of generality, we may assume that $\|x_i^*\|_0 = 1$ and $\|x_i\| \le C < \infty$, where $\|\cdot\|_0$ is the original norm on X. We introduce on the space X* the new norm $\|\cdot\|$, setting for each $x^* \in X^*$

$$\|x^*\| = \max\left\{\frac{C}{C+1}\|x^*\|_0,\ \sup_i \|P_i^*(x^*)\|_0\right\}.$$

Clearly, the norm $\|\cdot\|$ is equivalent to the original norm on X*. It is easily verified that the unit sphere of the space $(X^*, \|\cdot\|)$ is closed in the topology of $\sigma(X^*, X)$. Thus the space $(X^*, \|\cdot\|)$ is conjugate to the space $(X, \|\cdot\|)$, where $\|\cdot\|$ is some equivalent norm to the original norm on X. Let Y be a Banach space such that Y* is isometric to $(X^*, \|\cdot\|)$. Denote by π_X the natural imbedding of X in X**, and by π_Y the natural embedding of Y in X** (we are identifying the spaces X* and Y*). Since Y is preconjugate to X*, $r(\pi_Y(Y), X^*) = 1$, and by Corollary 1 $\pi_X(X) \subseteq \pi_Y(Y)$. We show that in fact we have equality $\pi_X(X) = \pi_Y(Y)$. This will also mean that the spaces X and Y are isometric. Since X is preconjugate to X*, $X^{***} = X^* \oplus (\pi_X(X))^{\perp}$. If $\pi_X(X) \subset \pi_Y(Y)$, then $(\pi_Y(Y))^{\perp} \subset (\pi_X(X))^{\perp}$, and thus $X^* \oplus (\pi_Y(Y))^{\perp} \neq X^{***}$, i.e., $Y^* = X^{***}/(\pi_Y(Y))^{\perp} \neq X^*$; this contradiction completes the proof of the theorem.

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CAPACITY AND CONTINUATION OF FUNCTIONS WITH GENERALIZED DERIVATIVES

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In this article we continue our study of the conditions on the boundary of a region [1-3], for which it is possible to continue functions in Sobolev's classes over the boundary of the region, preserving class. This problem was considered for regions with sufficiently smooth boundary by Nikol'skii [4], Babich [5], and for regions with Lipshitz boundary by many authors: Calderon [6], Solntsev [7], Smith [8], Besov and Il'in [9-11], Burenkov [12-13], and others. A detailed bibliography can be found in the above-mentioned articles.

The main aim of this article is to obtain necessary and sufficient conditions on the boundary of a planar region, so that it and its complement simultaneously admit continuation of functions in the classes L_p^1 (W_p^1) over the boundary of the region. The region is assumed to be bounded and simply-connected.

A necessary and sufficient condition on the boundary of a region as in the case of the space L_2^1 (W_2^1) [1] or B_p^l ($l < 1$, $lp = 2$) [2-3] is the condition obtained by Ahlfors [14], who was studying the continuation of quasiconformal homeomorphisms of planar regions. In the planar case, this condition coincides with the necessary and sufficient condition for continuation in the class BV, obtained by Burago and Maz'ya [15], if we require that the condition is satisfied simultaneously for the region and its complement.

The method of proof we shall use is connected with the study of the behavior of the capacity induced by the space L_p^l. We present this in such a way as to select the small number of properties of the capacity (not depending a priori on the nature of the space) which give necessary conditions for continuation. We shall prove these properties of the capacity only for the spaces L_p^l.

The study of sufficient conditions is connected with the invariance of the spaces L_p^l for quasi-isometries [16].

For a region in n-dimensional Euclidean space R^n ($n > 2$), the results we obtain remain applicable, but as a set of separate necessary conditions and a set of separate sufficient conditions, which do not coincide.

1. FORMULATION OF THE BASIC RESULTS. PROOF OF SUFFICIENCY

1.1. Ahlfors' Condition for a Nonbounded Planar Simply Connected Region [14]

Let the boundary γ of the bounded simply connected planar region G be a Jordan curve. The region G satisfies Ahlfors' condition, if for any triple of points $\xi_1, \xi_2, \xi_3 \in \gamma$ we have the inequality $|\xi_3 - \xi_1| < C|\xi_1 - \xi_2|$, if the point ξ_3 lies between the points ξ_1 and ξ_2. The constant $C > 1$ does not depend on the choice of the triple ξ_1, ξ_2, ξ_3.

THEOREM 1.1 [14]. Let the boundary γ of a nonbounded simply connected region G be a Jordan curve satisfying Ahlfors' condition. Then there exists a homeomorphism $\varphi : \bar{G} \to R^2 \backslash G$ fixed on the boundary, i.e., $\varphi(x) = x$ for all $x \in \gamma$, differentiable inside the region, and satisfying for all $x, y \in G$ the inequalities

$$\frac{1}{P(C)} \leqslant \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leqslant P(C)$$

where the constant P(C) depends only on the constant in Ahlfors' condition.

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