BANACH SPACES WITHOUT COMPLETE MINIMAL SYSTEMS

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A system of elements $\{x_i\}_{i\in I}$ of a Banach space X is called minimal if each element of the system does not belong to the closed linear hull of the remaining ones. This is equivalent to saying that there exists a system of linear functionals $\{x_i^*\}_{i\in I} \subset X^*$ (called the conjugate of the original system) such that $x_i^*(x_j) = \delta_{ij}$. If the linear hull of system $\{x_i\}_{i\in I}$ is dense in X ($[x_i] = X$), then system $\{x_i\}_{i\in I}$ is called complete. A Banach space X is called a Grothendieck space if in X* weak* and weak convergence of sequences coincide (see [1]). "Typical" examples of Grothendieck spaces are the so-called injective spaces and, in particular, the space m(S) of functions bounded on the set S with the uniform norm (see [1]). It is known [2] that a nonreflexive Grothendieck space cannot have a complete minimal system, whose conjugate is a total system. In this note we prove that certain Grothendieck spaces cannot have any complete minimal systems.

Let τ be a cardinal number. We denote by $m_\tau(S)$ the subspace of m(S) consisting of those functions x(s) for which the power of the support $S(x) = \{s\in S : x(s) \neq 0\}$ does not exceed τ. Let T be a subset of S. We identify the space $m_\tau(T)$ with the (complemented) subspaces of $m_\tau(S)$ formed of those functions whose supports lie in T. For a normed space X we denote by dens X its weight, i.e., the minimal power of a set dense in X.

LEMMA. Let $X = m_\tau(S)$ and $\mathrm{Card}\, S > \tau$, and let Y be a subspace of X such that the factor-space $E = X/Y$ is reflexive. Then there is a subset $T \subset S$ such that $\mathrm{Card}\, T < \tau$ and $m_\tau(S \setminus T) \subset Y$.

Proof. We suppose that the assertion does not hold. Since $Y \neq m_\tau(S)$ there is an element $x_1 \in X$ of unit norm that does not belong to Y. We set $T_1 = S(x_1)$ and $R_1 = S \setminus T_1$. Since $\mathrm{Card}\, T_1 \leqslant \tau < \mathrm{Card}\, S$, by our assumption $m_\tau(R_1) \not\subset Y$. Hence we can find an element $x_2 \in m_\tau(R_1) \setminus Y$, $\|x_2\| = 1$. We set $T_2 = T_1 \cup S(x_2)$ and $R_2 = S \setminus T_2$. Since $\mathrm{Card}\, T_2 \leqslant \tau < \mathrm{Card}\, S$, we can find an $x_3 \in m_\tau(R_2) \setminus Y$, $\|x_3\| = 1$. By continuing this process indefinitely, we obtain an uncountable family of elements $\{x_i\} \not\subset Y$ of norm 1 whose supports are pairwise disjoint: $S(x_i) \cap S(x_j) = \varnothing$ $(i \neq j)$. Because the family $\{x_i\}$ is uncountable, it contains a subsequence $\{x_{i_n}\}$ such that the norms of the images of elements under the factor-mapping $F : X \to E$ are uniformly separated from $\|F(x_{i_n})\| \geqslant \delta > 0$ for all n = 1, 2, ... Since $\{x_{i_n}\}_{n=1}^\infty$ is normalized and disjoint, it is equivalent to the natural basis in the space c_0. The operator $A = F|_{[x_{i_n}]}$ is a continuous linear operator mapping a space isometric to c_0 into a reflexive space E. Hence, A is a compact operator. Since the sequence $\{x_{i_n}\}_{n=1}^\infty$ converges weakly to the zero element (as the natural basis for c_0), the sequence of images $\{A(x_{i_n})\}_{n=1}^\infty$ must converge to zero in the norm topology of E. But $\|A(x_{i_n})\| = \|F(x_{i_n})\| \geqslant \delta > 0$ for all n = 1, 2, ..., and so we reach a contradiction.

THEOREM 1. If $\mathrm{Card}\, S > 2^\tau$, then the space $m_\tau(S)$ does not have complete minimal systems.†

Proof. We suppoee that despite the assertion of the theorem, the space $X = m_\tau(S)$ has a complete minimal system $\{x_i\}_{i\in I}$. We can show that $m_\tau(S)$ is isomorphic to the space C(Q), where C is a certain σ-Stonian compact set; then it follows that $m_\tau(S)$ is a Grothendieck space (see [3] and [4]). Therefore, by a theorem of Johnson [1] the space $\Gamma = [x_i^*]$ must be reflexive. We denote by $Y = \Gamma_\perp$ the annihilator of Γ in X and set $E = X/Y$. Since a reflexive subspace of a dual space is weak* closed (see [4, Proposition 1]), $\Gamma = (\Gamma_\perp)^\perp = Y^\perp = E^*$. Hence, the space E is reflexive, because its dual is reflexive. By the lemma there is a set $T \subset S$ such that $\mathrm{Card}\, T \leqslant \tau$ and $m_\tau(S \setminus T) \subset Y$. Let P be the natural projection from $m_\tau(S)$ onto $m_\tau(T) = m(T) : P(x) = x|_T$. For each $i \in I$ we set $z_i = P(x_i)$. Since $x_i - z_i \in m_\tau(S \setminus T) \subset Y$, we have $x_j^*(z_i) = x_j^*(x_i) - x_j^*(x_i - z_i) = x_j^*(x_i) = \delta_{ij}$ for all $i, j \in I$. Consequently, $\{z_i\}_{i\in I}$ is a minimal system in the space m(T). Hence, $\mathrm{Card}\, I = \mathrm{Card}\, \{z_i\}_{i\in I} \leqslant \mathrm{Card}\, m(T) = 2^{\mathrm{Card}\, T} \leqslant 2^\tau$. On the other hand, since the system $\{x_i\}_{i\in I}$ is complete in $m_\tau(S)$, we have $\mathrm{Card}\, I \geqslant \mathrm{Card}\, S$. Thus $\mathrm{Card}\, I \leqslant 2^\tau < \mathrm{Card}\, S \leqslant \mathrm{Card}\, I$. This contradiction completes the proof of the theorem.

†In the case when $\tau = \aleph_0$ this result was proved somewhat earlier by A. N. Plichko.

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COROLLARY. If $X = m_\tau(S)$ and $\mathrm{Card}\, S > 2^\tau$, then every factor-space of X of weight dens X does not have complete minimal systems.

In fact, as Levinson and Johnson have remarked (see [5, Remark 5]), the availability of a complete minimal system in a Grothendieck space Z is equivalent to the existence in Z of a reflexive factor-space of weight dens Z. Let E be some factor-space of X of weight dens X. Since the space E* is identified with a weak* closed subspace in X*, E is a Grothendieck space. If we assume that E has a complete minimal system, then it must have a reflexive factor-space E_1 of weight dens E_1 = dens E = dens X. It is easy to see that E_1 is a factor-space of X, and so X must have a complete minimal system.

We can treat the space m(S) in an obvious way as a space of continuous function on the Stone–Čech compactification of the set S. It is known that the space m(S) has a complete biorthogonal bounded system (i.e., $\sup \|x_i\| \cdot \|x_i^*\| \leqslant C < \infty$; see [5]). The next theorem shows that these systems in the space m(S) have a certain "strange" property.

THEOREM 2. Let X = C(K) be a Grothendieck space and $\{x_i\}_{i \in I}$ be a complete minimal bounded system in X. Then every subsequence of $\{x_i\}_{i\in I}$ contains a subsequence equivalent to the natural basis of the space l_1.

Proof. We assume that $\|x_i\| = 1$ and $\|x_i^*\| \leqslant C$ $(i \in I)$. Since X is a Grothendieck space the space $\Gamma = [x_i^*] \subset X^*$ is reflexive and its unit ball U is weakly compact. We consider an arbitrary subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ of the system $\{x_i\}_{i\in I}$, and assume that it is weakly fundamental. Then the sequence $\{x_{i_{2k}} - x_{i_{2k-1}}\}_{k=1}^{\infty}$ converges weakly to the zero element, and by condition for the weak compactness of bounded sets in the space C(K)* (see [6]), this convergence to the zero element must be uniform on the set U:

$$\lim_{k\to\infty} \sup_{f \in U} |f(x_{i_{2k}} - x_{i_{2k-1}})| = 0.$$

But the system $\{x_i\}_{i\in I}$ is minimal and so for any k = 1, 2, ... we have

$$\sup_{f\in U} |f(x_{i_{2k}} - x_{i_{2k-1}})| \geqslant \frac{1}{C} |x^*_{i_{2k}}(x_{i_{2k}} - x_{i_{2k-1}})| = \frac{1}{C}.$$

Thus, no subsequence of system $\{x_i\}_{i\in I}$ can be weakly fundamental. The conclusion of the theorem now follows from the well-known Rosenthal characterization of spaces containing l_1 (see [7]).

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