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# TWO COUNTER-EXAMPLES IN NONSEPARABLE BANACH SPACES

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It is shown that the well-known theorem of Kadec for the $H_\Gamma$ renorming of separable Banach spaces, when $\Gamma$ is a norming subspace in the dual, cannot be extended to the class of nonseparable Banach spaces.

## 1. INTRODUCTION

Let $X$ be a Banach space and let $\Gamma$ be a total subspace in the dual space $X^*$.

The norm $\|.\|$ on a Banach space $X$ is said to have the *$H_\Gamma$-property*, if for sequences on the unit sphere, $\sigma(X, \Gamma)$ and norm convergence coincide, that is whenever $x_0, x_n \in X$ $(n < \infty)$, $\lim_n \|x_n\| = \|x_0\|$ and $\lim_n f(x_n) = f(x_0)$ for all $f \in \Gamma$, then $\lim_n \|x_n - x_0\| = 0$.

The norm $\|.\|$ on a Banach space $X$ is said to have the *$K_\Gamma$-property*, if the $\sigma(X, \Gamma)$ and norm topologies coincide on the unit sphere.

Obviously, if the norm has the $K_\Gamma$-property, then it has the $H_\Gamma$-property. The converse is not true.

When $\Gamma = X^*$, then the $H_{X^*}$-property is known as the *H-property* or the *Kadec-Klee property* and the $K_{X^*}$-property is known as the *Kadec property*.

If the Banach space $X$ admits an equivalent norm with the $H_\Gamma$ or $K_\Gamma$ property, then we write $X \in (H_\Gamma)$ or $X \in (K_\Gamma)$.

It is easy to see that, if $\Gamma$ is a total separable subspace in $X^*$, then $X \in (K_\Gamma)$ if and only if $X \in (H_\Gamma)$.

The following result of Kadec [5] is well-known: Let $X$ be a separable Banach space and $\Gamma$ is a norming subspace in $X^*$. Then $X \in (H_\Gamma)$.

Naturally, the question arises: Can we extend this result to the class of nonseparable Banach spaces?

The answer to this question is negative.

Plicko proved in [7] that, if $\Gamma$ is a total subspace in $X^*$ such that $\operatorname{dens}(\Gamma) < \operatorname{dens}(X)$, then $X \notin (K_\Gamma)$. In particular, it follows that, if $X$ is a separable Banach space with a nonseparable dual space $X^*$, then $X^* \notin (H_X)$.

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Here, we give two examples of total subspaces $\Gamma$ in $X^*$, for some concrete Banach spaces $X$, with $\operatorname{dens}(\Gamma) = \operatorname{dens}(X)$, such that $X \notin (H_\Gamma)$.

We denote by $\overline{\text{lin}}(A)$ the closed linear hull of the set $A \subset X$; $\operatorname{dens}(X)$ is the density character of $X$, that is, the smallest cardinal for which $X$ has a dense subset of the same cardinality.

Let $\Gamma$ be a subspace of $X^*$. We say that $\Gamma$ is *norming* if its Dixmier characteristic

$$r(\Gamma) = \inf_{\|x\|=1} \sup_{f\in\Gamma} \frac{|f(x)|}{\|f\|} > 0.$$

**LEMMA.** *Let $(X, \rho_1)$ be an uncountable separable metric space, $(Y, \rho_2)$ be a separable metric space and $T\colon X \to Y$ be an arbitrary map. Then there exist point $x_0 \in X$ and sequence $\{x_n\}_{n<\infty}$ in $X$, $x_n \neq x_0$, $\forall n < \infty$, such that $\lim_n \rho_1(x_n, x_0) = 0$ and $\lim_n \rho_2(Tx_n, Tx_0) = 0$.*

In this case, we say that $x_0$ is a *point of partial continuity* for the map $T$.

## 2. First example

Let $AP$ be the Banach space of all almost periodic functions defined on the real line $\mathbb{R}$ with the supremum norm $\|.\|_\infty$.

We define the linear functionals $\delta_t \in AP^*$ for every $t \in \mathbb{R}$ by the equality $\delta_t(f) = f(t)$, $f \in AP$, and define the subspace $\Gamma = \overline{\text{lin}}(\delta_t)_{t\in\mathbb{R}}$ in $AP^*$.

The subspace $\Gamma$ is norming.

Really, if $f \in AP$, $\|f\|_\infty = 1$, then there exists a sequence $\{t_n\}_{n<\infty} \subset R$ such that $\lim_n |f(t_n)| = 1$, that is $\lim_n |\delta_{t_n}(f)| = 1$. Since $\sup_{\delta\in\Gamma}(|\delta(f)| / \|\delta\|) \geqslant |\delta_{t_n}(f)|$, $\forall n < \infty$, then $r(\Gamma) = 1$.

**PROPOSITION 1.** *The space $AP \notin (H_\Gamma)$.*

PROOF: Let $f_\lambda(t) = e^{i\lambda t}$, $\lambda \in \mathbb{R}$, and let $\|.\|$ be an equivalent norm on Banach space $AP$.

We examine the function $\lambda \mapsto \|f_\lambda\|$, $\lambda \in \mathbb{R}$.

According to the Lemma, there exists a point of partial continuity for this function, that is, there exist $\lambda_0, \lambda_n \in \mathbb{R}$, $\lambda_n \neq \lambda_0$ $(n < \infty)$ such that

$$\lim_n \lambda_n = \lambda_0, \tag{1}$$

$$\lim_n \|f_{\lambda_n}\| = \|f_{\lambda_0}\|. \tag{2}$$

From (1) and the definition of the functions $f_{\lambda_n}$ we get $\lim_n f_{\lambda_n}(t) = f_{\lambda_0}(t)$, $\forall t \in \mathbb{R}$, which is equivalent to

$$\lim_n \delta_t(f_{\lambda_n}) = \delta_t(f_{\lambda_0}), \quad \forall t \in \mathbb{R}. \tag{3}$$

Consequently, from (3) for all $\delta \in \Gamma$ we have

$$\lim_n \delta(f_{\lambda_n}) = \delta(f_{\lambda_0}). \tag{4}$$

Now, if we suppose that the space $AP \in (H_\Gamma)$ then from (2) and (4) it follows that $\lim_n \|f_{\lambda_n} - f_{\lambda_0}\| = 0$ which is, obviously, impossible, since the system $\{f_\lambda\}_{\lambda \in \mathbb{R}}$ is minimal (see [6]). The proposition is proved. □

## 3. Second example

Let $QC[0, 1]$ be the Banach space of all real-valued functions defined on $[0, 1]$ for which $f(t+0) = f(t)$ for every $t$, that is, $f$ is continuous from the right and $f(t-0)$ exists for every $t$ with the supremum norm $\|.\|_\infty$.

Let $E$ be a dense subset in $[0, 1]$ such that the set $E_1 = [0, 1] \setminus E$ is uncountable.

We define the linear functionals $\delta_t \in QC[0, 1]^*$ for every $t \in E$ by the equality $\delta_t(f) = f(t)$, $f \in QC[0, 1]$, and define the subspace $\Gamma = \overline{\mathrm{lin}}(\delta_t)_{t \in E}$ in $QC[0, 1]^*$.

The subspace $\Gamma$ is norming.

Really, if $f \in QC[0, 1]$, $\|f\|_\infty = 1$, then there exists a sequence $\{t_n\}_{n<\infty} \subset E$ such that $\lim_n |f(t_n)| = 1$, that is, $\lim_n |\delta_{t_n}(f)| = 1$. Since $\sup_{\delta \in \Gamma} (|\delta(f)| / \|\delta\|) \geqslant |\delta_{t_n}(f)|$, $\forall n < \infty$, then $r(\Gamma) = 1$.

**Proposition 2.** *The space $QC[0, 1] \notin (H_\Gamma)$.*

Proof: For every $s \in E_1$ we define the functions $f_s(t) = 0$, if $0 \leqslant t < s$ and $f_s(t) = 1$, if $s \leqslant t \leqslant 1$. Obviously, $f_s \in QC[0, 1]$ for all $s \in E_1$. Let $\|.\|$ be an equivalent norm on the Banach space $QC[0, 1]$.

We examine the function $s \mapsto \|f_s\|$, $s \in E_1$.

According to the Lemma, there exists a point of partial continuity for this function, that is, there exist $s_0, s_n \in E_1$, $s_n \neq s_0$ $(n < \infty)$ such that

$$\lim_n s_n = s_0, \tag{5}$$

$$\lim_n \|f_{s_n}\| = \|f_{s_0}\|. \tag{6}$$

From (5) and definition of the functions $f_{s_n}$ we get $\lim_n f_{s_n}(t) = f_{s_0}(t)$, $\forall t \in E$, which is equivalent to

$$\lim_n \delta_t(f_{s_n}) = \delta_t(f_{s_0}), \quad \forall t \in E. \tag{7}$$

Consequently, from (3) for all $\delta \in \Gamma$ we have

$$\lim_n \delta(f_{s_n}) = \delta(f_{s_0}). \tag{8}$$

Now, if we suppose that the space $QC[0, 1] \in (H_\Gamma)$ then from (6) and (8) it follows that $\lim_n \|f_{s_n} - f_{s_0}\| = 0$ which is, obviously, impossible, since $\|f_{s_n} - f_{s_0}\| = 1$ for all $n < \infty$. The proposition is proved. □

REMARKS. (i) The spaces $AP$ and $QC[0, 1]$ possess an equivalent locally uniformly convex norm and, consequently, they have the Kadec property (see [**1**, **2**, **3**]).

(ii) Godun proved in [4] that, if $(x_i, f_i)_{i\in I}$ is $M$-basis in the Banach space $X$ and $\Gamma = \overline{\mathrm{lin}}(f_i)_{i\in I}$, then $X \in (H_\Gamma)$ if and only if the subspace $\Gamma$ is norming.

This gives rise to the following.

QUESTION. *Let $X$ be a nonseparable Banach space and let $\Gamma$ be a norming subspace in dual the space $X^*$. What sufficient conditions must $\Gamma$ satisfy so that $X$ admits the $H_\Gamma$-property?*

## REFERENCES

[1] G.A. Alexandrov, *Locally uniformly convex equivalent norms in nonseparable Banach spaces*, (in Russian), Ph.D. Dissertation: Kharkov, 1980.

[2] G.A. Alexandrov and K.-D. Kursten, 'Some nonseparable Banach spaces which admit equivalent locally uniformly convex norms', in Russian, *Izv. Vyssh. Ucheben. Zaved. Mat.* **7** (1981), 10.

[3] G.A. Alexandrov, 'Equivalent locally uniformly convex renorming of nonseparable Banach spaces', in Russian, in *Constructive function theory 81 (Varna 1981)* (Bulgar. Acad. Sci., Sofia, 1983), pp. 9–13.

[4] B.V. Godun, 'On the Markushevich bases', in Russian, *Dokl. Acad. Nauk SSSR* **266** (1982), 11–14.

[5] M.I. Kadec, 'On the connection between weak and strong convergence', in Ukraninian, *Dopovidi Acad. Nauk Ukrain. RSR* **9** (1957), 949–952.

[6] L.H. Loomis, *An introduction to abstract harmonic analysis* (D. van Nostrand Co, Toronto, New York, London, 1953).

[7] A.N. Plicko, 'On the projective resolution of identity, the Markushevich bases and the equivalent norms', in Russian, *Mat. Zametki* **34** (1983), 719–726.

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