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MATHEMATIQUES
Analyse fonctionnelle

AN INFINITE DIMENSIONAL SUBSPACE OF $C[0,1]$ CONSISTING OF NOWHERE DIFFERENTIABLE FUNCTIONS

V. P. Fonf[1], V. I. Gurariy*, M. I. Kadets[2]**

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Questions about differentiability and relative properties of functions which form an infinite dimensional subspace of $C[0,1]$ attracted the attention of several mathematicians (see [2-6]). In particular, in [2] an infinite dimensional linear (nonclosed) manifold in $C[0,1]$ is constructed whose only anywhere differentiable element is the function $x(t) \equiv 0$. In 1990 the authors constructed an infinite dimensional (closed) subspace E of $C[0,1]$ (actually E isomorphic to l_1) having the following properties:
(1) each function from E (but the zero function) is nowhere differentiable
and
(2) each non-zero function from E has neither right nor left finite derivative almost everywhere (in the sense of Lebesgue measure).

This result,which has been circulated as a preprint, was partially generalized by L. RODRIGUEZ-PIAZZA in [6]. Namely, using a different technique, he proves that every separable Banach space is linearly isometric to a closed subspace X of the space of continuous functions on $[0,1]$, such that every non-zero function in X is nowhere differentiable.

However, as it is proved by S. BANACH [1], almost all functions in $C[0,1]$ (in the sense of Baire category) have neither left nor right (finite) derivative at any point of $[0,1]$. And so the following question is natural.

Question. Is there an infinite dimensional closed subspace $Y \subset C[0,1]$ such that each non-zero function in Y has neither left nor right finite derivative at any point of $[0,1]$?

Thus the space E still holds some interest, since it gives a partial answer on the question.

Theorem. There exist a closed infinite dimensional subspace E of $C[0,1]$ and a subset A of $[0,1]$ of Lebesgue measure one, such that if $f \in E$ and f is non-zero function, then f is nowhere differentiable on $[0,1]$ and has neither right nor left finite derivative on the set A.

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Proof. Our basic functions are analogous to the Van der Waerden's function. Let

$$u_1(x) = \begin{cases} x & \text{if } 0 \le x \le 1/4\,; \\ 1/2 - x & \text{if } 1/4 \le x \le 3/4\,; \\ x - 1 & \text{otherwise}\,, \end{cases}$$

and extend periodically to R with period 1. Further, let $u_n(x) = 8^{1-n}u_1(8^{n-1}x)$, $n = 2, 3, \ldots$. This sequence of functions has a number of properties. First the number 8^{1-n} is the common period of the functions $u_n(x), u_{n+1}(x), \ldots$. The extreme value of the functions $u_p(x)$, $p = 1, 2, \ldots n$ are taken at

$$1/(4 \cdot 8^{p-1}) \;+\; k/(2 \cdot 8^{p-1}), \quad k = 0, 1, \ldots;\; p = 1, \ldots n,$$

which are not internal to the intervals

$$J_{n,s,l} = [\frac{l + (s-1)/4}{8^{n-1}}\,,\, \frac{l + s/4}{8^{n-1}}]\,,\; s = 1, 2, 3, 4;\; l = 0, 1, \ldots 8^{n-1} - 1.$$

In particular, on each of these intervals each of the functions $u_p(x)$, $p = 1, \ldots n$, is linear and for any points x_1, x_2 in the interval $J_{n,s,l}$ we have

$$\frac{u_p(x_2) - u_p(x_1)}{x_2 - x_1} = \pm 1,\; p = 1, \ldots n.$$

Set $\sigma_p = \{2^{p-1} \cdot n :\; n \in N\}$ and $\varphi_p(x) = \sum_{n \in \sigma_p} u_n(x)$, $(p = 1, \ldots n)$. The sequence $\{\varphi_p/\|\varphi_p\|\}_{p=1}^{\infty}$ in $C[0,1]$ is equivalent to the unit basis of l_1. (This is proved analogously to the corresponding statement for a lacunary trigonometric system c.f. [3].) Let $E = [\varphi_p]_{p=1}^{\infty}$ be the closed subspace spanned by $\{\varphi_p\}_{p=1}^{\infty}$ and let $\psi(x)$ be a non-zero function in E. Given a point x_0 in $[0,1]$ we now show that $\psi(x)$ has no derivative at x_0. Since $\{\varphi_p\}_{p=1}^{\infty}$ is a basis for E we can write $\psi = \sum_{i=1}^{\infty} a_i\varphi_i$ or more precisely $\psi = \sum_{i=q}^{\infty} a_i\varphi_i$ where $q = \min\{i :\; a_i \neq 0\}$. Let us assume the existence of a finite derivative of ψ at x_0 denoted $\psi'(x_0)$. We now choose j big enough to insure 1) $j > q$ and 2) if $|x - x_0| < 8^{-2^j}$ and $|x' - x_0| < 8^{-2^j}$ then

$$|\frac{\psi(x) - \psi(x_0)}{x - x_0} - \frac{\psi(x') - \psi(x_0)}{x' - x_0}| < |a_q|/2. \tag{1}$$

For $n = 2^{j-1}$ we can find integers l and s, with $1 \le l \le 8^{n-1} - 1$ and $1 \le s \le 4$ such that $x_0 \in J_{n,s,l}$. Choose $x = x(n)$ such that $|x - x_0| = 8^{-n}$ and the interval connected the points x and x_0 is contained in $J_{n,s,l}$. Now we make the remark that will be essential for the proof of the second part of the theorem:

Remark. If in the base 8 the expansion x is $x = 0.p_1, p_2, \ldots$ then for an even integer p_n $x(n) > x_0$ and for an odd integer p_n $x(n) < x_0$.

Let us consider the ratio

$$\frac{\psi(x) - \psi(x_0)}{x - x_0} = \sum_{i=q}^{\infty} a_i \sum_{r \in \sigma_i} \frac{u_r(x) - u_r(x_0)}{x - x_0} =$$

$$\sum_{i\geq q, 2^{i-1}\leq n} a_i \sum_{1\leq t\leq m_i, m_i 2^{q-1}\leq n} \frac{u_{t2^{i-1}}(x) - u_{t2^{i-1}}(x_0)}{x - x_0} =$$

$$\sum_{i\geq q, 2^{i-1}\leq n} a_i \sum_{1\leq t\leq m_i, m_i 2^{i-1}\leq n} \pm 1.$$

(We recall that 8^{-n} is a period of the functions $u_{n+1}(x), u_{n+2}(x), \ldots$ and that $\min \sigma_i = 2^{i-1}$). For $n' = n + 2^{q-1}$ we can find integers l', s' with $1 \leq l' \leq 8^{n'-1}$ and $1 \leq s \leq 4$ such that $x_0 \in J_{n',s',l'}$. Set $x' = x(n')$ to denote a point such that $|x_0 - x'| = 8^{-n'}$ and the interval connected the points x_0 and x' is contained in $J_{n',s',l'}$. It is easy to see that

(2) $$J_{n',s',l'} \subseteq J_{n,s,l}.$$

Let us compare the ratio

$$\frac{\psi(x') - \psi(x_0)}{x' - x_0} = \sum_{i=q}^{\infty} a_i \sum_{r\in\sigma_i} \frac{u_r(x') - u_r(x_0)}{x' - x_0} =$$

$$\sum_{i\geq q, 2^{i-1}\leq n'} a_i \sum_{1\leq t\leq m_i', m_i' 2^{i-1}\leq n'} \frac{u_{t2^{i-1}}(x') - u_{t2^{i-1}}(x_0)}{x' - x_0} =$$

$$\sum_{i\geq q, 2^{i-1}\leq n+2^{q-1}} a_i \sum_{1\leq t\leq m_i', m_i' 2^{i-1}\leq n+2^{q-1}} \pm 1$$

with the ratio

$$\frac{\psi(x) - \psi(x_0)}{x - x_0}.$$

The number of summands in the exterior sums for both ratios is the same (for the first ratio: $2^{i-1} \leq n = 2^{j-1}$ and so $i \leq j$ and for the second ratio $2^{i-1} \leq 2^{j-1} + 2^{q-1} < 2^{j-1} + 2^{j-1} = 2^j, (q < j!)$, and so $i \leq j$). The number of summands in each interior sum starting with the second is the same too (for each $i \geq q + 1$, for the first ratio $m_i 2^{i-1} \leq n$ that is $m_i 2^{i-1} \leq 2^{j-1}$ and so $m_i = 2^{j-1}$ and for the second ratio $m_i' 2^{i-1} \leq 2^{j-1} + 2^{q-1} < 2^{j-1} + 2^{i-1}$, that is $m_i' < 2^{j-1} + 1$, or $m_i' = 2^{j-1}$). With the help of (2) one can deduce that corresponding summands in every interior sum starting with the second coincide for both ratios. Consider now the first interior sums. For the first ratio $m_q 2^{q-1} \leq 2^{j-1}$ or $m_q \leq 2^{i-q}$. For the second ratio $m_q' 2^{q-1} \leq 2^{j-1} + 2^{q-1}$ or $m_q' = 2^{i-q} + 1$. Thus, the difference between the first interior sum for the second ratio and the first interior sum for the first ratio is equal to ± 1. That is

$$\left|\frac{\psi(x) - \psi(x_0)}{x - x_0} - \frac{\psi(x') - \psi(x_0)}{x' - x_0}\right| = |a_q|,$$

which contradicts (1) and hence completes the proof of the first part of the theorem.

Let us now consider the question about left and right derivatives of functions in E. For each integer q let

$$D_q = \{n \in N : \; n = 2^{j-1} + k2^{q-1}, \; j \in N; \; k = 0, 1.\} = \{n_i^{(q)}\}_{i=1}^{\infty},$$

where the elements of $\{n_i^{(q)}\}_{i=1}^{\infty}$ are arranged in increasing order. For each point $x \in [0,1]$ we denote by x_p the p-th digit in the base 8 expansion of x, that is $x = \sum_{p\geq 1} x_p 8^{-p}$. Let A_q be the set of points $x \in [0,1]$ such that for every natural number l there exists $s > l$ such that

$$x_{n_s^{(q)}} = x_{n_{s+1}^{(q)}} = x_{n_{s+2}^{(q)}} = 1$$

and

$$x_{n_{s+3}^{(q)}} = x_{n_{s+4}^{(q)}} = x_{n_{s+5}^{(q)}} = 2.$$

The standard argument shows that the Lebesgue measure $m(A_q) = 1,\ (q = 1, 2, \ldots)$ and, therefore, $m(\cap_{q=1}^{\infty} A_q) = 1$. For $A = \cap_{q=1}^{\infty} A_q$ after repeating the arguments, given in the first part of the proof and taking into account the remark, we can establish that each $u(x) \in E$ has neither left nor right derivative at any point $x \in A$. The theorem is proved.

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Department of Mathematics
and Computer Science
Ben-Gurion University of the Negev
P.O,B. 653, Beer-Sheva 84105, Israel
e-mail: `fonf@black.bgu.ac.il`

* *Department of Mathematics*
and Computer Science
Kent State University
P.O.B. 5190 Kent OH 44242-0001, USA

** *Department of Mathematics*
KHDAMG, Revolution Str. 12
310002 Kharkov, Ukraine