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ON PERMUTATIONS OF BIORTHOGONAL DECOMPOSITIONS

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SOMMARIO. — Nell'articolo diamo applicazioni del classico teorema di Steinitz all'investigazione di permutazioni di sistemi biortogonali in spazi di Banach. Inoltre viene studiata una versione debole della nota base di Terenzi-Revesz. Alcuni problemi aperti vengono posti.

The main purpose of this paper is to show how the well known theorem of Steinitz (generalization of Riemann theorem about the permutations of series of real numbers) can work in the theory of the biorthogonal decompositions.

Let us recall this theorem:

THEOREM OF STEINITZ [1]. - "Let Σu_n be a series in R^m, which we consider Euclidean space with the natural inner product $\langle ... \rangle$, then the series converges if an only if the following two conditions are satisfied

(a) $\lim_{n \to \infty} u_n = 0$,

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(b) for every $v \in R^m$, if $\Sigma \mid \langle u_n, v\rangle \mid$ is divergent, then also $\Sigma\ [\langle u_n, v\rangle]^+$ and $\Sigma \mid \langle u_n, v\rangle]^-$ are divergent

(where, if an $\{a_n\} = \{a_{n_k}\} \cup \{a_{n'_k}\}$ with $a_{n_k} \geqslant 0$ and $a_{n'_k} < 0$ for every k, then $\Sigma\,[a_n]^+$ means $\Sigma\, a_{n_k}$ and $\Sigma\,[a_n]^-$ means $\Sigma\, a_{n'_k}$).

If these conditions are satisfied the domain of the sums of the series (i.e. the set of sums of all its convergent permutations) has the form $u + V_1$, where u is sum of one of the convergent permutations, while the "convergent subspace" V_0 (i.e. the set of all $v \in R^m$ for which $\Sigma \mid \langle u_n, v\rangle \mid < +\infty$) is orthogonal complement to V_1, hence $R^m = V_0 \oplus V_1$.

The direct cause of this paper was a sudden recent result of S. Revesz [2]: "For every continuous 2π-periodical function there exist a permutation of the associated Fourier series and a sequence of partial sums which is uniformly convergent to the function".

Let $\{f_n, e_n\}_1^\infty$ be a basis of Markushevich (M-basis) of a Banach space X, that is a biorthogonal system ($e_n \in X$, $f_n \in X^*$, $f_i(e_j) = \delta_{ij}$) with $\{\varrho_n\}$ fundamental in X and $\{f_n\}$ total on X. We can compare every element $x \in X$ with the associated biorthogonal decomposition

$$x \sim \Sigma\, f_n(x)\, e_n\,.$$

For an arbitrary not empty finite set $\mathcal{J} = \{n_1, \ldots, n_m\}$ of indices we define the partial sum of decomposition

$$S_{\mathcal{J}}(x) = \sum_{j=1}^{m} f_{n_j}(x)\, e_{n_j}\,;$$

we denote by $\mathfrak{F}(x)$ the set of all possible $S_{\mathcal{J}}(x)$.

We call $\{e_n\}_1^\infty$ a *basis of Terenzi-Revesz* (*TR-basis*) if for every $x \in X$ there exists a permutation of the biorthogonal decomposition

$$x \sim \sum_{1}^{\infty} f_{\pi(n)}(x)\, e_{\pi(n)}$$

so that a subsequence of partial sums is convergent to x. These "bases with individual brackets and permutation" were firstly considered in [3], see also [4].

We can reformulate the result of Revesz in terms of partial sums: every 2π-periodical continuous function belongs to the closure of $\mathfrak{F}(x)$ in the metric of $C_{(0,\pi)}$; this is corollary of the next simple proposition:

PROPOSITION 1. - *The fundamental minimal system* $\{e_n\}_1^\infty$ *in* X *is TR-basis if and only if every* $x \in X$ *belongs to the closure of* $\mathfrak{F}(x)$.

PROOF. - Necessity is obvious. For the sufficiency let us choose a positive sequence $\{\epsilon_k\} \longrightarrow 0$ and let us consider

$$x_1 = \sum_{i=1}^{N_1} f_{n_{i,1}}(x)\, e_{n_{i,1}}, \quad \| x - x_1 \| < \epsilon_1 .$$

For the element $h_1 = x - x_1$ we have $f_{n_{i,1}}(h_1) = 0$ for $1 \leqslant i \leqslant N_1$, hence there exists

$$x_2 = \sum_{i=1}^{N_1} f_{n_{i,2}}(x)\, e_{n_{i,2}} \text{ such that } \| h_2 \| < \epsilon_2 ,$$

where $h_2 = h_1 - x_2$, moreover $\{n_{i,1}\}_{i=1}^{N_1} \cap \{i,2\}_{i=1}^{N_2} = \phi$.

We have $x = x_1 + h_1 = x_1 + x_2 + h_2$.

So proceeding we obtain the necessary arrangement of indices and brackets

$$e_{n_{1,1}}, \ldots, e_{n_{N_1,2}}, \ldots, e_{n_{1,2}}, \ldots, e_{n_{N_2,2}}, \ldots ;$$

the proof is finished.

For the general M-basis, if we do not give further conditions, we cannot expect the existence of some method to restore the element of the space only by means of partial sums of its decomposition.

For example it is possible to have the following situation: the numerical series, obtained after the application of a linear functional f to the decomposition of an element x, has only a finite number of nonzero items; however the sum of this series is not $f(x)$.

This situation, described in I. Singer's monography [5], is connected with the "non hereditary completeness".

We remind that a M-basis $\{x_n\}$ is called *hereditary complete* [6] if

$$x \in V\,(f_k(x)\,x_k,\ k \geqslant 1) \quad \text{for every } x\,,$$

where $V(K)$ means the closed linear hull of a set K; such system is also called *strong M-basis* [5].

Let us consider some conditions on an M-basis $\{e_n\}_1^\infty \subset X$ with coniugate system $\{f_n\}_1^\infty \subset X^*$.

CONDITION (0): $\forall x,\ f,\ \lim\limits_{n\to\infty} f_n(x)\,f(e_n) = 0.$

CONDITION (I):

(a) If for x and f we have $\sum\limits_{i=1}^{\infty} |f_n(x)\,f(e_n)| < +\infty$, then $\sum\limits_{n=1}^{\infty} f_n(x)\,f(e_n) = f(x)$.

(b) If for x and f we have $\sum\limits_{n=1}^{\infty} |f_n(x)\,f(e_n)| = +\infty$, then $\sum\limits_{1}^{\infty} [f_n(x)\,f(e_n)]^+ = +\infty$ and $\sum\limits_{1}^{\infty} [f_n(x)\,f(e_n)]^- = +\infty$.

CONDITION (II): For every x and f there exists a permutation π of the natural sequence so that the permuted series

$$\sum_{1}^{\infty} f_{\pi(n)}(x)\,f(e_{\pi(n)}) \quad \text{is convergent to } f(x)\,.$$

(In this case we call $\{e_n\}^\infty$ *basis of Steinitz* or *S-basis*).

CONDITION (III): Every element x belongs to the closure of the convex hull of the set $\mathfrak{F}(x)$.

PROPOSITION 2. - *Condition* (II) *is equivalent to the union of the conditions* (0) *and* (I).

It is a simple corollary of Riemann's theorem about the permutations of a numerical series.

PROPOSITION 3. - *Conditions* (I) *and* (III) *are equivalent.*

PROOF. (I) $\longrightarrow$ (III). - Otherwise suppose that there exists x so that $x \notin \overline{\text{conv}}\ \mathfrak{F}(x)$;
by Hahn-Banach theorem there exists a linear functional f such that

$$f(x) > \sup \{f(y);\ y \in \text{conv}\ \mathfrak{F}(x)\} . \tag{a}$$

Suppose now that $\sum_1^\infty f_n(x) f(e_n) = f(x)$, it follows that $\lim_{n\to\infty} f(S_n x) = f(x)$ which is contrary to (a), hence we cannot have I (a). But also I (b) is not possible, because $\sum_1^\infty [f_n(x) f(e_n)]^+ = +\infty$ is contrary to (a); therefore $\overline{(\text{III})} \Longrightarrow \overline{(\text{I})}$.

Let us show that (III) $\Longrightarrow$ (I):

We can imagine condition (III) in the following form: for every $x \in X$ and $\epsilon > 0$ there exist a natural number N and a set of coefficients $\{\lambda_{N,K}\}_{k=1}^N$, $0 \leqslant \lambda_{N,k} \leqslant 1$ such that

$$\| x - \sum_1^N \lambda_{N,k} f_k(x) e_k \| < \epsilon . \tag{b}$$

Suppose that for some non zero x and f we have

$$\sum_1^\infty |f_k(x) f(e_k)| = C < +\infty$$

Fix $\epsilon > 0$. Let n be a natural number such that

$$\sum_{k=n+1}^\infty |f_k(x) f(e_k)| < \epsilon/3 .$$

By (b) there exist a natural number $N \geqslant n$ and $\{\lambda_{N,K}\}_{k=1}^N$ such that

$$\| x - \sum_1^N \lambda_{N,k} f_k(x) e_k \| < \frac{\epsilon}{3\|f\|}, \quad 1 - \frac{\epsilon}{3C} \leqslant \lambda_{N,k} \leqslant 1 \text{ for } 1 \leqslant k \leqslant n .$$

$0 \leqslant \lambda_{N,k} \leqslant 1$ for $n+1 \leqslant k \leqslant N$.

Then we have that

$$|f(x) - \sum_{1}^{n} f_k(x) f(e_k)| \leqslant |f(x) - \sum_{1}^{N} \lambda_{N,k} f_k(x) f(e_k)| + \\ + \sum_{1}^{n} (1-\lambda_{N,k}) f_k(x) f(e_k)| + |\sum_{n+1}^{N} \lambda_{N,k} f_k(x) f(e_k)| \leqslant \\ \leqslant \|f\| \cdot \frac{\epsilon}{3\|f\|} + \frac{\epsilon}{3C} \cdot C + \frac{\epsilon}{3} = \epsilon .$$

Consequently (III) $\Longrightarrow$ (Ia).

Suppose now that

$$\sum_{1}^{\infty} |f_k(x) f(e_k)| = +\infty .$$

From the inequality

$$\epsilon \|f\| \geqslant |f(x) - \sum_{1}^{N} \lambda_{N,k} f_k(x) f(e_k)| \geqslant |\sum_{1}^{N} \lambda_{N,k} [f_k(x) f(e_k)]^+ | + \\ - \sum_{1}^{N} \lambda_{N,k} [f_k(x) f(e_k)]^- | - |f(x)| ,$$

since $\lim_{N\to\infty} \lambda_{N,k} = 1$ for every k, we obtain that $\sum_{1}^{\infty} [f_k(x) f(e_k)]^+$ and $\sum_{1}^{\infty} [f_k(x) f(e_k)]^-$ are divergent; that is (III) $\Longrightarrow$ (Ib). Proposition 3 is proved.

COROLLARY 1. - *Every summing on Cesaro basis $\{e_n\}_1^\infty$ in X is also S-basis in X. Hence the trigonometrical system is S-basis in the space of 2π-periodical continuous functions.*

Next theorem is the main result of the paper.

THEOREM. - *Let $\{e_n\}_1^\infty$ be S-basis in X. Then for every $x \in X$ and for every linearly independent finite set $\{h_\mu\}_1^m \subset X^*$, there exists a permutation of the natural sequence $\pi : N \longrightarrow N$ such that*

$$\sum_{n=1}^{\infty} f_{\pi(n)}(x)\, h_\mu(e_{\pi(n)}) = h_\mu(x), \quad \text{for } 1 \leqslant \mu \leqslant m .$$

PROOF. - Let us consider the linear map $T : X \longrightarrow R^m$ where we compare every $x \in X$ with the m-dimensional vector $Tx = \{h_\mu(x)\}_1^m$ (we remind that R^m is the Euclidean space with the natural inner-product). Without restriction of generality we may account T surjective, then $T^* : R^m \longrightarrow X^*$ will be injective. Let us consider x with the biorthogonal decomposition

$$x \sim \sum_1^\infty f_n(x)\, e_n\,,$$

we can compare it with the series in R^m

$$\sum_1^\infty u_n = \sum_1^\infty \{f_n(x)\, h_\mu(e_n)\}_{\mu=1}^m\,.$$

By conditions (I)-(II) the series satisfies Steinitz's theorem, so we have convergent permutations and the domain of sums is $u + V_1$, moreover $R^m = V_0 \oplus V_1$ where V_0 is the subspace of convergence.

Let us consider for every $v \in R^m$ the numerical series

$$\sum_{n=1}^\infty \langle u_n, v\rangle\,.$$

If $v \in V_0$, according to condition (Ia), for every permutation π the series

$$\sum_1^\infty \langle u_{\mu(n)}, v\rangle \text{ converges to } \langle Tx, v\rangle\,.$$

On the other hand by Steinitz's theorem there exists a permutation π such that for every $v \in V_1$ the series $\sum_1^\infty \langle u_{\mu(n)}, v\rangle$ converges to $\langle Tx, v\rangle$.

That is for an arbitrary $x \in X$ there is a permutation π such that, for every $v \in R^m$, the series

$$\sum \langle u_{\pi(n)}, v\rangle \text{ is convergent to } \langle Tx, v\rangle\,.$$

So, for every h_μ with $1 \leqslant \mu \leqslant m$, if $T^* v_\mu = h_\mu$, the series $\sum \langle u_{\pi(n)}, v_\mu \rangle$ is convergent to $\langle Tx, v \rangle$, hence $\sum f_{\pi(n)} h_\mu (e_{\pi(n)})$ is convergent to $h_\mu (x)$; which proves theorem.

COROLLARY 2. - *Let $\{e_n\}_1^\infty$ be S-basis of X. Then every x of X belongs to the weak closure of the set of all partial sums of its biorthogonal decomposition.*

PROOF. - Let us set

$$0 (x) = \left\{ y \in X : | h_\mu (x-y) | < \epsilon, \ 1 \leqslant \mu \leqslant m \right\},$$

that is an arbitrary weak neighborhood of an element x.

By theorem there exists a partial sum of a permutated decomposition of x,

$$s = \sum_{k=1}^{n} f_{\pi(k)} (x) \, e_{\pi(k)} ,$$

so that $| h_\mu (x-s) | < \epsilon$ for $1 \leqslant \mu \leqslant m$; that is $s \in 0 (x)$. Corollary 2 is proved.

COROLLARY 3. - *A S-basis $\{e_n\}_1^\infty$ in X is a "non linear summing basis" in the following sense:*

for every $x \in X$ there exists a triangular matrix $\{\varrho_{ni} (x) : n \in N, 1 \leqslant i \leqslant n\}$ such that

$$0 \leqslant \varrho_{ni} (x) \leqslant 1, \quad \lim_{n \to \infty} \left\| x - \sum_{1}^{n} \varrho_{ni} (x) \, f_i (x) \, e_i \right\| = 0 .$$

PROOF. - Since x belongs to the weak closure of the set of the partial sums of its decomposition, by a known theorem of Mazur x belongs to the strong closure of the convex hull of this set. So for every $\epsilon > 0$ there exists a finite set of partial sums such that a convex combination ϵ-approaches x. Obviously every convex combination of partial sums has the form

$$\sum_{1}^{\infty} \varrho_i \, f_i (x) \, e_i, \qquad 0 \geqslant e_i \leqslant 1 .$$

The question of existence of the S-basis in every separable Banach space is open. In connection with this let us introduce the following definition.

DEFINITION. - *Let* $\{e_N\}$ *be M-basis in* X *and let us consider two sets* $A \subseteq X$, $B \subseteq X^*$. *We call* $\{e_n\}$ *weak basis with individual permutation (individual brackets) relatively* (A, B), *if for every pair* (x, f) $x \in A$, $f \in B$, *there exists a permutation* $\pi(n)$ *(a sequence of natural numbers* $n_1 < n_2 < \dots$*) such that*

$$f(x) = \sum_{1}^{\infty} f_{\pi(n)}(x)\, f(\varrho_{\pi(n)}) \quad \left(f(x) = \lim_{k\to\infty} \sum_{i=1}^{n_k} f_i(x)\, f(\varrho_i)\right).$$

In the case of $A = X$, $B = X^*$ let us call $\{e_n\}$ the *weak basis with individual permutation* (*weak basis with individual brackets*); in this line let us also define the *weak* (*TR*)-*basis* (i.e. with individual brackets and permutation simultaneously).

Now questions of existence arise for the mentioned weak bases for different (A, B); for example in the case when A is the subset of the continuously differentiable functions in C, $\{e_n\}$ is fundamental in C and orthonormal system in L^2.

Obviously $\{e_n\}$ is weak basis with individual permutations if and only if it is S-basis.

It is interesting to construct examples of weak bases of one type, which are not weak bases of another type; similarly to examples of V.M. Kadets [4].

The following hypothesis seems verosimilar:

HYPOTHESIS. - If an M-basis $\{e_n\}$ in X is uniformly minimal and also weak basis with individual brackets, then it is also weak basis with individual permutations, i.e. S-basis.

We remind that the system $\{x_k\}$ in X is called uniformly minimal if there exists $a > 0$ such that

$$(*) \qquad \varrho(x_k, L^{(k)}) \geqslant a \, \|x_k\|, \qquad k = 1, 2, \dots,$$

where ϱ is the distance and $L^{(k)} = V(x_1, \dots, x_{k-1}, X_{k+1}, \dots)$.

PROPOSITION 4. - *If a weak (TR)-basis* $\{e_n\}$ *satisfies condition* (0), *then it is S-basis.*

PROOF. - It is sufficient to check that conditions (Ia) and (Ib) are satisfied:

(Ia) If for some x, f it is

$$\sum_1^{\infty} |f_n(x) f(e_n)| < \infty,$$

there exists a permutation π and an arrangement of brackets so that

$$\lim_{k \to \infty} \sum_{i=1}^{N_k} f_{\pi(i)}(x) f(e_{\pi(i)}) = f(x),$$

then the absolutely convergent series $\sum_1^{\infty} f_n(x) f(e_n)$ can converge only to $f(x)$.

(Ib) If for some x, f it is

$$\sum_1^{\infty} |f_n(x) f(e_n)| = +\infty,$$

then obviously $\sum_{m=1} [f_n(x) f(e_n)]^+ = \infty$ and $\sum_{n=1} [f_n(x) f(e_n)]^- = \infty$, otherwise if one of these sums would be finite, $\{e_n\}$ would not be weak (TR)-basis. Proposition 4 is proved.

PROBLEM 1. - Does there exist a weak TR-basis which is not S-basis? Indeed we have the following property.

PROPOSITION 5. - *Every uniformly minimal TR-basis is S-basis.*

PROOF. - By above it is only sufficient to check condition (0), for a system $\{f_n, e_n\}$, where for simplicity we can suppose that

$$\|e_n\| = 1, \quad \|f_n\| < C < \infty, \qquad n = 1, 2, \ldots$$

(since after we can use the proof of proposition 4).

(0) For an arbitrary $x \in X$ there exists a sequence of natural numbers $n_1 < n_2 < \ldots$ such that, if for simplicity we write the items in the natural numeration, we have

$$\lim_{k \to \infty} \sum_{i=1}^{n_k} f_i(x)\, e_i = x\,,$$

so $\lim_{k \to \infty} \sum_{i=n_k+1}^{n_{k+1}} f_i(x)\, e_i = 0$. We affirm that $f_i(x) \longrightarrow 0$; indeed otherwise there would exist a sequence of natural numbers $\{m_j\}$, such that, for every j,

$$f_{m_j}(x) \geqslant b > 0\,,$$

hence, if $n_{k_j+1} \leqslant m_j \leqslant n_{k_j+1}$, by (*) we would have

$$\Big\| \sum_{i=n_{k_j+1}}^{n_{k_j+1}} f_i(x)\, e_i \Big\| \geqslant ab\,,$$

that is a contradiction.

Hence for every $f \in X^*$ $|f_i(x)\, f(e_i)| \geqslant |f_i(x)| \cdot \|f\| \xrightarrow[i \to \infty]{} 0$; proposition 5 is proved.

Problem 2. - Does in every separable Banach space exist the weak TR-basis?