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Countability of the Spectrum of a Weakly Almost-Periodic Function with Values in a Banach Space*

M. I. Kadets and K. D. Kyursten

A function $F(t)$, defined on the real axis $\mathbf{R}$ and taking values in a Banach space X, is weakly almost-periodic (w.a.p.) if the scalar-valued function $\langle x^*, F(t)\rangle$ is an almost-periodic Bohr function for every linear functional $x^* \in X^*$. Many authors have studied weakly almost-periodic functions, on the one hand in connection with the theory of differential equations and, on the other hand, as independent objects (see [**1**] and [**3**]).

By the spectrum of a w.a.p. function $F(t)$ we shall mean the union of the spectra of all almost-periodic functions $\langle x^*, F(t)\rangle$ when x^* runs through X^* (in fact, it is enough that x^* runs through a subset which is dense in X^* with respect to the norm topology). Countability of the spectra of w.a.p. functions was previously established only for certain classes of Banach spaces (X^* separable, X weakly sequentially complete, X separable and conjugate). However, for an arbitrary Banach space the question of the countability of spectra of w.a.p. functions has remained open.

The aim of the present note is to prove countability of the spectrum of a w.a.p. function in the general case. The image of a w.a.p. function is separable and therefore we may restrict our considerations to a separable Banach space X. The space of all almost-periodic Bohr functions with the natural norm is denoted by AP.

Given a w.a.p. function $F(t)$ let us define, with its help, linear bounded operators V and T acting according to the formulas

$$Vx^* = \langle x^*, F(t)\rangle; \qquad \langle x^*, T\psi\rangle = \int_{-\infty}^{\infty} \langle x^*, F(t)\rangle \varphi(t)\, dt; \qquad \varphi \in L^1(\mathbf{R}).$$

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By the definition of a w.a.p. function, V is an operator from X^* to AP. The operator T is defined in $L^1(\mathbf{R})$ and its image is a subset of X^{**}. We shall show that T actually maps $L^1(\mathbf{R})$ into X.

Lemma 1. *If a function $F(t)$: $\mathbf{R} \to X$ is weakly continuous and bounded then the linear operator T defined by the formula*

$$\langle x^*, T\varphi \rangle = \int_{-\infty}^{\infty} \langle x^*, F(t) \rangle \varphi(t)\, dt; \qquad \varphi \in L^1(\mathbf{R})$$

maps $L^1(\mathbf{R})$ into X.

Proof. The function $F(t)$ maps every compact interval $[a, b] \subset \mathbf{R}$ to a weakly compact subset K of X. The values of the "truncated" operator T_n:

$$\langle x^*, T_n\varphi \rangle = \int_{-n}^{n} \langle x^*, F(t) \rangle \varphi(t)\, dt$$

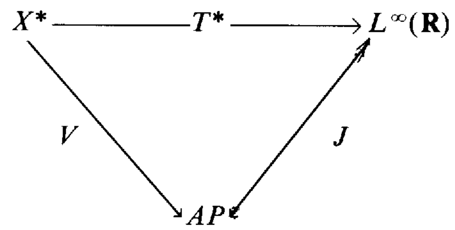
for $\|\varphi\| \leq 1$ are contained in the weakly closed convex hull of K and, consequently, are contained in X. In this way, T_n maps $L^1(\mathbf{R})$ into X. Since $F(t)$ is bounded on the real axis, $\|F(t)\| \leq A < \infty$, we get

$$\left| \int_{-\infty}^{-n} + \int_{n}^{\infty} \langle x^*, F(t) \rangle \varphi(t)\, dt \right| \leq \|x^*\| \cdot A \cdot \left[\int_{-\infty}^{-n} + \int_{n}^{\infty} |\varphi(t)|\, dt \right],$$

and passage to the limit from $T_n\varphi$ to $T\varphi$ does not lead out of X:

$$\lim_{n \to \infty} \|T_n\varphi - T\varphi\| = 0; \qquad \varphi \in L^1(\mathbf{R}).$$

Thus, starting from a w.a.p. function $F(t)$ we have defined two bounded linear operators V: $X^* \to AP$ and T: $L^1(\mathbf{R}) \to X$, which are related to each other by the commutative diagram



where J is the natural imbedding from AP into $L^\infty(\mathbf{R})$.

Let B^* be the unit ball of X^*. Then its image $K = T^*B^*$ is a weakly* compact convex subset of $L^\infty(\mathbf{R})$. We may identify K and VB^* ($K = JVB^*$) and say that

K is a convex subset of $AP \subset L^\infty(\mathbf{R})$ and compact in the topology $\sigma = \sigma(L^\infty(\mathbf{R}), L^1(\mathbf{R}))$. Note that this topology is metrizable on K.

Lemma 2. *A subset N of AP is separable if and only if the union of the spectra of all almost-periodic functions in N is at most countable.*

Proof. It is well known that the required union of spectra is obtained by taking, not the entire set N, but only a dense subset. This means that the separability of N implies the countability of the union of spectra. Conversely, if the union of spectra is countable then N is contained in the separable subspace of AP spanned by the exponential functions $\exp(i\lambda t)$, where λ runs through the union of spectra.

Theorem 1. *The spectrum of a weakly almost-periodic function with values in a Banach space is at most countable.*

Proof. By Lemma 2 we need to show that the set $K \subset AP \subset L^\infty(\mathbf{R})$ is separable in the strong topology. Let us assume that K is not separable. We choose a subset Δ of K which has two properties: a) in the topology σ the set Δ is homeomorphic to the countable product of two-point sets D^ω (the Cantor discontinuum) and b) every uncountable subset of Δ is nonseparable in the norm topology. An analogous construction was made in **[4]**.

Let μ be the measure in Δ induced by the Haar measure of the group D^ω. Let us consider the Banach space $L^1(\Delta, \mu)$ and the bounded linear operator $S: L^1(\Delta, \mu) \to L^\infty(\mathbf{R})$ defined by the formula

$$\langle S\psi, \varphi \rangle = \int_\Delta \langle f, \varphi \rangle \psi(f)\, d\mu(f); \qquad \psi \in L^1(\Delta, \mu); \qquad \varphi \in L^1(\mathbf{R}).$$

In fact, S maps $L^1(\Delta, \mu)$ into $AP \subset L^\infty(\mathbf{R})$, since Δ is contained in the weakly compact convex subset K of AP. Let us denote the strong closure of $SL^1(\Delta, \mu)$ in AP by G. Since $L^1(\Delta, \mu)$ is separable, G is separable. The union of the spectra of all almost-periodic functions in G is denoted by Λ. Then Λ is countable. Let M be the smallest module that contains Λ and H the separable subspace of AP consisting of the functions whose Fourier exponents are in M.

Let us introduce some linear operators associated with the Bochner-Fejér summation method for the Fourier series of almost-periodic functions. Let $(K_m(t))_1^\infty$ be the sequence of Bochner-Fejér kernels realizing the summation of the Fourier series of the functions of H. For every m, let us define a sequence

$P_{mn}, n = 1, 2, \ldots,$ of linear operators in $L^1(\mathbf{R})$:

$$\langle f, P_{mn}\varphi \rangle = \int_{-\infty}^{\infty} \left[\frac{1}{2n} \int_{-n}^{n} K_m(t-\tau) f(\tau)\, d\tau \right] \varphi(t)\, dt; \qquad f \in L^\infty(\mathbf{R}). \qquad (1)$$

Clearly, for a fixed $\varphi \in L^1(\mathbf{R})$, the identity $\langle f, P_{mn}\varphi \rangle \equiv \langle P^*_{mn} f, \varphi \rangle$ defines a function of f, which is continuous with respect to the topology σ. Let us pass to the limit in (1) as $n \to \infty$, where we consider that $f \in AP$ (otherwise convergence cannot be guaranteed):

$$\lim_{n\to\infty} \langle P^*_{mn} f, \varphi \rangle = \langle P_m f, \varphi \rangle; \qquad f \in AP, \ \varphi \in L^1(\mathbf{R}). \qquad (2)$$

If we now pass to the limit as $m \to \infty$, we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \langle P^*_{mn} f, \varphi \rangle = \langle Pf, \varphi \rangle, \qquad (3)$$

where the operator P, as follows from the properties of the Bochner-Fejér method, is a projector with norm one and projects AP onto H. Now let $f \in \Delta \subset AP$. For every fixed $\varphi \in L^1(\mathbf{R})$, the functions $\langle f, P_{mn}\varphi \rangle$ are continuous on Δ with respect to the topology σ, and the functions $\langle P_m f, \varphi \rangle$ belong to the first Baire class while $\langle Pf, \varphi \rangle$ is in the second Baire class, and in addition, all of them are bounded. Making use of the limit relations (2) and (3) we obtain the following identity:

$$\int_\Delta \langle Pf - f \rangle \psi(f)\, d\mu(f) \equiv 0; \qquad \psi \in L^1(\Delta, \mu). \qquad (4)$$

Evidently,

$$\begin{aligned} \int_\Delta \langle Pf, \varphi \rangle \psi(f)\, d\mu(f) &= \int_\Delta \lim_{m,n} \langle f, P_{mn}\varphi \rangle \psi(f)\, d\mu(f) \\ &= \lim_{mn} \int_\Delta \langle f, P_{mn}\varphi \rangle \psi(f)\, d\mu(f) \\ &= \lim_{m,n} \langle S\psi, P_{mn}\varphi \rangle \\ &= \langle PS\psi, \varphi \rangle. \end{aligned}$$

The identity (4) is proved, since $S\psi \in G \subset H$ and therefore $RS\psi = S\psi$ in Δ.

It follows from (4) that $Pf = f$ on a subset Δ_1 of Δ with $\mu(\Delta_1) = \mu(\Delta)$. This subset is uncountable and, by property b) in the definition of Δ, is nonseparable. On the other hand, the image of P is separable. Here we have a contradiction. This proves the theorem.

The countability of the spectrum of a w.a.p. function established in Theorem 1 permits us to extend the approach associated with the compactification of the real axis to the case of w.a.p. functions. Let $F(t)$ be a w.a.p. function, $\Lambda = (\lambda_k)$

its spectrum and M the smallest module that contains the spectrum. With the help of this module we can, in the usual manner for the theory of almost-periodic functions, equip the real axis $\mathbf{R}$ with a topology with respect to which it becomes a precompact metrizable group Ω_M; its completion is denoted by T_M. On Ω_M the function $F(t)$ is weakly uniformly continuous (i.e., every function $\langle x^*, F(t)\rangle$, $x^* \in X^*$, is uniformly continuous). By continuity $F(t)$ can be extended to T_M. But now it is possible that its values on $T_M \setminus \Omega_M$ are not contained in X (they are contained in the weak* sequential closure of X in X^{**}). Within the framework of this approach the following results are easily obtained.

Corollary 1. *The image of a w.a.p. function is metrizable with respect to the weak topology of X.*

Corollary 2. *A weakly continuous function $F(t)$ is weakly almost-periodic if and only if every sequence (s_n) in $\mathbf{R}$ has a subsequence (t_n) such that the sequence of shifted functions $(F(t + t_n))$ is weakly fundamental uniformly in $\mathbf{R}$.*

Remark 1. A function $F(t)\colon \mathbf{R} \to X$ is called a w.a.p. Besicovitch function if for every $x^* \in X^*$ the function $\langle x^*, F(t)\rangle$ is a Besicovitch almost-periodic function (see [2], p. 142). As it turns out, Theorem 1 cannot be extended to Besicovitch functions.

Example 1. Let us define $F(t) = \exp(ixt)$, $t \in \mathbf{R}$, $x \in [0, 1]$. It can be shown that $F(t)$ is a w.a.p. Besicovitch function with values in $C[0, 1]$ and with non-denumerable spectrum (its spectrum covers the interval $[0, 1]$).

Remark 2. The method applied in the proof of Theorem 1 is also applicable in other situations. For example, the following result can be proved.

Theorem 2. *In the conjugate space C^* of the Banach space $C = C[0, 1]$ let E be the closed subspace formed by all purely atomic measures. If a weakly compact convex subset K of C^* is contained in E, then it is separable.*

References

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