REPRINTED FROM

SYMPOSIUM ON INFINITE DIMENSIONAL TOPOLOGY

EDITED BY

R. D. ANDERSON

ANNALS OF MATHEMATICS STUDIES, 69

PRINCETON, NEW JERSEY

1972

ON TOPOLOGICAL CLASSIFICATION OF

NON-SEPARABLE BANACH SPACES

C. BESSAGA AND M. I. KADEC

The problem of topological classification of separable Banach spaces has been completely solved; any such a space, if infinite-dimensional, is homeomorphic to the Hilbert space $l_2(\aleph_0)$ [17]. The problem, if every (non-separable) Banach space is homeomorphic to a Hilbert space, is still open. However in a few interesting cases (for instance for reflexive spaces [5]) the affirmative answer has been established. This supports the conjecture that every Banach space is homeomorphic to a Hilbert space, i.e., that the only topological invariant of Banach spaces is their density character.

Most of the facts on topological equivalence of Banach spaces have been obtained by combining so-called "coordinate methods" and "decomposition methods" with some result of linear character. These methods have found also some applications in case of linear metric spaces which are not Banach spaces: [4], [20], see also [6], [7].

The sections 1 and 2 are devoted to describing coordinate methods in an ordinal number set-up (generalization of "separable" coordinate methods: [3], [13] – [16], [19], [20]) and to outlining their applications. In section 3 we state main theorems of decomposition type and list the results which can be obtained by help of these theorems.

Notation. The letters: α, β, γ, τ, ν denote ordinal numbers; the first ordinal number of a cardinality $\aleph$ will be also denoted by $\aleph$. e, ξ, t denote real numbers. x, y, z denote vectors. All the Banach spaces considered here are over the real scalars. $c_0(\nu)$ is the Banach space of all $x = \{\xi_\alpha\}_{\alpha<\nu}$ such that the set $\{\alpha: |\xi_\alpha| > \varepsilon\}$ is finite for every $\varepsilon > 0$, under the norm $\|x\| = \sup_\alpha |\xi_\alpha|$. $l_p(\nu)$, $p \geq 1$, is the space of all $x = \{\xi_\alpha\}_{\alpha<\nu}$ such that the set $\{\alpha: \xi_\alpha \neq 0\}$ is at most countable and $\|x\| = (\Sigma_\alpha |\xi_\alpha|^p)^{1/p}$. In the sequel, the space $l_1(\nu)$, denoted briefly $l(\nu)$, is in a sense a "test space"; of course it could be replaced by the Hilbert space $l_2(\nu)$, because the latter space is homeomorphic to the first under the map $\{\xi_\alpha\} \to \{\xi_\alpha|\xi_\alpha|\}$, see also Theorem 1.

The symbol " $\simeq$ " denotes the relation of being homeomorphic.

1. *Bernstein maps:* B*-systems and* co-B*-systems*

Let X be a Banach space, and let b be a δ-*modular* on X, i.e., a continuous non-negative functional defined on X such that: $\lim_n b(x_n) = 0$ iff $\lim_n x_n = 0$ and $b(tx)$ is non-increasing in t for every $x \in X$, $t \geq 0$. A closed subspace L of X is called a b-*Čebyšev subspace* provided that for any x in X there is the unique b-*nearest point* $Px \in L$, i.e., Px has the property: $z = Px$ iff $z \in L$ and $b(x - z) = \inf\{b(x-y): y \in L\}$.

Any system $\langle b, \{L_\alpha\}_{\alpha\leq\nu}\rangle$, where b is a δ-modular on X and L_α are b-Čebyšev subspaces of X, will be called a *generalized* [co-] T-*system* iff $L_1 = \{0\}$, $L_\nu = X$, $L_\beta \subset L_\alpha$ for $\alpha > \beta$, $L_\alpha = \overline{\cup_{\beta<\alpha} L_{\beta+1}}$ for $\alpha \leq \nu$ $\dim(L_{\alpha+1}/L_\alpha) = 1$ for $\alpha < \nu$ [$L_1 = X$, $L_\nu = \{0\}$, $L_\beta \supset L_\alpha$ for $\alpha > \beta$, $L_\alpha = \cap_{\beta<\alpha} L_{\beta+1}$, $\dim(L_\alpha/L_{\alpha+1}) = 1$].

An *oriented* generalized [co-] T-system

$$(*) \qquad \langle b, \{L_\alpha\}_{\alpha\leq\nu}, \{f_\alpha\}_{\alpha<\nu}\rangle$$

is a generalized [co-] T-system together with a family of linear functionals $f_\alpha \in L^*_{\alpha+1}$ [$f_\alpha \in L^*_\alpha$] such that $f_\alpha(x) = 0$ iff $x \in L_\alpha$ [$x \in L_{\alpha+1}$].

Let P: $X \to L_\alpha$ be the b-nearest-point map and let $d_\alpha(x) = b(x - P_\alpha x)$, $\varepsilon_\alpha(x) = \operatorname{sgn} f_\alpha(P_\alpha x)$ [$\varepsilon_\alpha(x) = -\operatorname{sgn} f_\alpha(P_{\alpha+1} x)$]. The system (*) induces the Bernstein map h: $X \to l(\nu)$, where $hx = (d_\alpha(x) - d_{\alpha+1}(x)) \cdot \varepsilon_\alpha(x)$.

An oriented generalized [co-] T-system will be called a *generalized* [co-] B-*system,* provided that h is a homeomorphism onto $l(\nu)$. Omitting the adjective "generalized" will indicate that $b(x) = \|x\|$, the norm of the space.*

It follows directly from the definition of generalized [co-] B-systems, that in order to establish a homeomorphism between a Banach space X and a space $l(\nu)$ it suffices to find in X either a generalized B-system or a generalized co-B-system. To check if a given generalized [co-] T-system is a generalized [co-] B-system is, in general, not difficult. For instance, in the countable case (more precisely if $\nu = \aleph_0$) every oriented T-system the Bernstein map of which is one-to-one, is a B-system (because all oriented T-systems, with $\nu = \aleph_0$, have the property that h is continuous, onto $l(\aleph_0)$ and h^{-1} preserves precompactness).

Example A. Let X be either $c_0(\nu)$ or $l_p(\nu)$, $p \geq 1$. Let $L_\alpha = \{\{\xi_\tau\} \in X: \xi_\alpha = 0 \text{ for } \tau \geq \alpha\}$, $L^\alpha = \{\{\xi_\tau\} \in X: \xi_\tau = 0 \text{ for } \tau < \alpha\}$, $f_\alpha(\{\xi_\tau\}) = \xi_\alpha$. Then:

1) $<\|\cdot\|, \{L_\alpha\}, \{f_\alpha\}> [<\|\cdot\|, \{L^\alpha\}, \{f_\alpha\}>]$ is a [co-] B-system in each $l_p(\nu)$.

2) There exists an equivalent norm $|||\cdot|||$ in $c_0(\nu)$ such that $<|||\cdot|||, \{L_\alpha\}, \{f_\alpha\}>$ is a B-system in $c_0(\nu)$ —Troyanski [24], cf. [13], [10]. This norm can be defined by Day's [11] formula: $|||x||| = \|x\| + \Sigma_n 2^{-n} |\xi_{p_n}|$, where $\{\xi_{p_n}\}$ is the sequence of all non-zero coordinates of x ordered in their non-increasing way.

Example B. Let X be a separable conjugate Banach space, $X = Y^*$. Let $\{y_n\}$ be a linearly independent sequence in Y such that $x(y_n) = 0$ for all n implies $x = 0$ and $\lim_k x_k(y_n) = x(y_n)$ for $n = 1, 2, \ldots$, $\lim_k \|x_k\| = \|x\|$ imply $\lim_k \|x_k - x\| = 0$. Denote $L^n = \{x: x(y_i) = 0 \text{ for } i < n\}$, $f_n(x) = x(y_n)$. Then $<\|\cdot\|, \{L^n\}, \{f_n\}>$ is a co-B-system. Using this argument it is possible to show that every separable conjugate Banach space admits general-

* The term "T-system" has been introduced by Klee and Long [20] (T for Tchebysheff). B-systems were called there "T-systems with Bernstein property."

ized co-B-systems, and therefore is homeomorphic to $l(\aleph_0)$; see [15], [16], [20].

By Example A we obviously have

THEOREM 1. *The spaces* $c_0(\nu)$ *and* $l_p(\nu)$, $p \geq 1$, *are homeomorphic.*

2. *B and co-B-systems related to projection bases*

Let X be a Banach space with a projection basis $\{S_\tau\}$. The symbols R_τ, e_τ, f_τ will have the same meaning as in [5]. We shall consider two systems of subspaces of X, $L_\alpha = S_\alpha X$ and $L^\alpha = R_\alpha X$.

Example C. X is either $c_0(\nu)$ or $l_p(\nu)$. $S_\alpha\{\xi_\tau\} = \{\xi'_\tau\}$, where

$$\xi'_\tau = \begin{cases} \xi_\tau & \text{for } \tau < \alpha \\ 0 & \text{for } \tau \geqq \alpha . \end{cases}$$

The subspaces L_α and L^α are now the same as in Example A.

A projection basis $\{S_\alpha\}_{\alpha \leq \nu}$ is said to be:

(a) *orthogonal,* iff L_α and L^α are norm-Čebyšev and S_α, R^α are the norm-nearest point maps onto L_α and L^α, respectively,

(b) *boundedly complete,* iff for any $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ and for any sequence $\{x_n\}$, with $x_n = (S_{\alpha_n} - S_{\alpha_{n+1}})x_n$ the condition $\sup_n \|\Sigma_{i=1}^n x_i\| < \infty$ implies the convergence of $\Sigma_n x_n$.

(c) *unconditional,* iff there exists a σ-additive projection-valued measure $E(\cdot)$ defined on the σ-field of all subsets of the segment $[1, \nu]$ such that $E\{\alpha\colon \alpha < \tau\} = S_\tau$, for all $\tau \leq \nu$.

PROPOSITION 1. If X has a projection basis, then it is possible to renorm X in such a way that the basis becomes orthogonal. (Cf. [5, Proposition 5].)

PROPOSITION 2. Every projection basis in a reflexive Banach space is boundedly complete.

PROPOSITION 3. Any reflexive Banach space contains a (reflexive) subspace of density character equal to that of the whole space, admitting a projection basis: [5].

THEOREM 2 *If* X *has a boundedly complete orthogonal projection basis* $\{S_\alpha\}$, f_α, R_α *and* L^α *are defined as above, and* $b(x) = \|x\|^2 + \Sigma_\alpha (\|S_{\alpha+1}x\| - \|S_\alpha x\|)\|R_{\alpha+1}x\|$, *then* $<b, \{L^\alpha\}, \{f_\alpha\}>$ *is a generalized co-B-system in* X. *Hence every Banach space with a boundedly complete projective basis of type* ν *is homeomorphic to* $l(\nu)$, *see* [5].

REMARK 1. In the case where the space X is uniformly convex, the above b(x) can be replaced by the norm of the space. S. Troyanski has shown that also in the case of boundedly complete unconditional bases one can use a norm in place of b(x).

Problem 1. Does every Banach space admit a generalized B-system?

Problem 2. Does every conjugate Banach space admit a generalized co-B-system?

REMARK 2. If X is separable and there exist closed bounded convex subsets of X without extreme points (for instance X = L(0, 1)), then X does not possess any co-B-system, see [8].

Problem 3. Let X be a Banach space with a projection basis $\{S_\alpha\}$. Does there exist a δ-modular b such that $<b, \{L_\alpha\}, \{f_\alpha\}>$ $[<b, \{L^\alpha\}, \{f_\alpha\}>]$ is a generalized [co-] B-system?

The problem is open even in the separable case; we have however:

THEOREM 3. *If* $\{S_\alpha\}_{\alpha \leq \nu}$ *is a projection basis in a Banach space* X, *then there exist an equivalent norm* $|||\cdot|||$ *and a* δ*-modular* b *such that* $<b, \{L^\alpha\}, \{f_\alpha\}>$ *is a co-T* system the Bernstein map of which restricted to the sphere* $\{x \in X: |||x||| = 1\}$ *is a homeomorphism onto the unit sphere of* $l(\nu)$.

(Take the functional F(x) defined in [18] and set

$$b(x) = \begin{cases} 1 - F(x) & \text{for } |||x||| \leq 1 \\ |||x||| & \text{for } |||x||| > 1 \end{cases}).$$

Problem 4. Let X be a Banach space with a basis $\{S_\alpha\}_{\alpha \leq \nu}$. Does there exist a new equivalent norm and a δ-modular b such that

$<b,\{L^{a}\},\{f_{a}\}>$ is a co-T.system and its Bernstein map restricted to the new unit sphere of X is a homeomorphism onto the unit sphere of $1(\nu)$?

More general:

Problem 5. Let X be a Banach space. Does there exist in X an oriented generalized [co-] T-system $<b,\{L_{a}\},\{f_{a}\}>$ such that $U = \{x: b(x) \leq 1\}$ is bounded and convex and the Bernstein map restricted to ∂U is a homeomorphism onto the unit sphere of $1(\nu)$?

The affirmative answer to this problem would imply $X \simeq 1(\nu)$.

3. *Decomposition theorems; applications*

Let X and Y be topological spaces. We shall write $Y | X$ provided that there exists a space W such that $X \simeq Y \times W$.

The next two theorems concern Fréchet spaces (i.e., locally convex complete linear metric spaces) and in particular are valid for Banach spaces:

THEOREM 4. *Let* X *and* Y *be Fréchet spaces. If either* Y *is a subspace of* X *or there is a continuous linear map* $T: X \xrightarrow[\text{onto}]{} Y$, *then* $Y | X$.

This is an easy corollary from Michael's Fréchet space version of Bartle-Graves result, see [22] and [2].

THEOREM 5. *Let* X *be a Fréchet space of density character* $\aleph$. *If* $1(\aleph) | X$ *then* $X \simeq 1(\aleph)$ [6, Th. 8.2].

COROLLARY 1. $(1(\aleph))^{\aleph_0} \simeq 1(\aleph)$.

From Propositions 3, 1, 2 and Theorems 2, 4, 5 it follows:

THEOREM 6. *Every reflexive Banach space is homeomorphic to a space* $1(\nu)$, see [5].

COROLLARY 2. *If* X *is a Banach space of density character* $\aleph$ *such that either* X *or* X^* *contains a reflexive subspace of density character* $\aleph$, *then* $X \simeq 1(\aleph)$ ([6], 9.3. xix).

The next theorem summarizes the known results on topological equivalence of spaces C(Q), of all continuous functions on the compact space Q, under the sup-norm:

THEOREM 7. *Let* Q *be a compact Hausdorff space. Each of the conditions (1)-(4) listed below is sufficient in order that* C(Q) *be homeomorphic with a space* $l(\nu)$:

(1) Q *is the one-point compactification of a discrete space,*

(2) Q *is the Stone-Cech compactification of a discrete space,*

(3) Q *is a topological group,*

(4) Q *contains a closed subset* D *such that* $C(D) \simeq l(\aleph)$, *where* $\aleph$ *is the density character of the space* C(Q).

REMARK 3. The condition (3) can be replaced by a weaker one:

(3′) Q admits a countable sequence of Baire measures μ_n such that the measure algebras $B(\mu_n, Q)$ are homogeneous in the sense of Maharam [21] and the supremum of density characters of these algebras is equal to the density character of the space C(Q).

The sufficiency of (1) follows from the Troyanski's result—Theorem 1 of this paper. The other condition has been established by Pełczynski [23], see also [6].

Problem 6. Let X be a C(Q) space with density character $\aleph \geq \aleph_0$. Must X be homeomorphic to the space $l(\aleph)$?

THEOREM 8. *Every abstract* L*-space is homeomorphic to a space* $l(\nu)$ ([6], 9.3 xx).

The above facts seem to suggest that the solution of the general classification problem of (non-separable) Banach spaces can be perhaps achieved by studying:

1) geometrical properties of Banach spaces connected with the existence of "nice" norms and δ modulars and "nice" generalized [co-] T-systems,

2) some isomorphic properties of Banach spaces, mainly the structure of subspaces and linear images of a given space.

We may expect also that the investigation of structural properties of Fréchet spaces will allow to reduce the classification problem of Fréchet spaces to that of Banach spaces. (In the separable case this was possible thanks to Anderson's theorem [1]: $l(\aleph_0) \simeq s$, the countable product of lines, Eidelheit's result [12] stating that every non-normable Fréchet space can be linearly mapped onto s, and Theorem 4.5.)

REFERENCES

[1] R. D. Anderson, "Hilbert space is homeomorphic to the countable infinite product of lines," *Bull. Amer. Math. Soc.*, 72 (1966), 515-519.

[2] R. G. Bartle and L. M. Graves, "Mappings between function spaces," *Trans. Amer. Math. Soc.*, 72 (1952), 400-413.

[3] S. Bernstein, "Sur le probléme inverse de la théorie de meilleure approximation des fonctions continues," *C. R. Acad. Sci. Paris, 206* (1938), 1520-1523.

[4] C. Bessaga, "Some remarks on homeomorphisms of $\aleph_0$-dimensional linear metric spaces," *Bull. Acad. Pol. Sci.*, ser. sci., math., astr. et phys., *11* (1963), 159-163.

[5] _____, "Topological equivalence of non-separable reflexive Banach spaces. Ordinal resolutions of identity and monotone bases," this issue p. 3.

[6] _____, "On topological classification of complete linear metric space *Fund. Math.*, *56* (1965), 251-288.

[7] _____ and A. Pełczynski, "Some remarks on homeomorphisms of F-spaces," *Bull. Acad. Pol. Sci.*, ser. sci., math., astr. et phys., *10* (1962), 265-270.

[8] _____, "On extreme points in separable conjugate spaces," *Israel J. Math.*, *4* (1966), 262-264.

[9] H. Corson and V. Klee, "Topological classification of convex sets," *Proc. Symp. Pure Math., 7,* "Convexity," *Amer. Math. Soc., 37-51.*

[10] R. O. Davis, "A norm satisfying the Bernstein condition," *Studia Math., 29* (1967), 219-220.

[11] M. M. Day, "Strict convexity and smoothness of normed spaces," *Trans. Amer. Math. Soc., 70* (1955), 516-528.

[12] M. Eidelheit, "Zur Theorie der Systeme linearen Gleichungen," *Studia Math., 6* (1936), 139-148.

[13] M. I. Kadec, "On homeomorphism of certain Banach spaces (Russian)," *Dokl. Akad. Nauk SSSR* (N. S.) *92* (1953), 465-468.

[14] _____, "On topological equivalence of uniformly convex spaces (Russian)," *Usp. Mat. Nauk, 10* (1955), 137-141.

[15] _____, "On connection between weak and strong convergence (Ukrainian)," *Dopovidi Akad. Nauk Ukrain., RSR 9* (1959), 465-468.

[16] _____, "On strong and weak convergence (Russian)," *Dokl. Akad. Nauk SSSR* (N. S) *122* (1958), 13-16.

[17] _____, "On topological equivalence of cones in Banach spaces (Russian)," *ibidem, 162* (1965), 1241-1244.

[18] _____, "On topological equivalence of all separable Banach spaces (Russian)," *ibidem, 167* (1966), 23-25.

[19] V. Klee, "Mappings into normed spaces," *Fund. Math., 49* (1960), 25-34.

[20] _____ and R. G. Long, "On a method of mapping due to Kadec and Bernstein," *Archiv. der Math., 8* (1957), 280-285.

[21] D. Maharam, "On homogeneous measure algebras," *Proc. Nat. Acad. Sci., USA, 28* (1942), 108-111.

[22] E. Michael, "Convex structure and continuous selections," *Canad. J. Math., 11* (1959), 556-576.

[23] A. Pełczynski, "Linear extensions and averagings of continuous functions," *Rozprawy Matematyczne,* in print.

[24] S. Troyanski, "On topological equivalence of $c_0(\aleph)$ and $l(\aleph)$ (Russian," *Bull. Acad. Polon. Sci.,* ser. sci., math., astr. et phys. (1967) 389-396.

Warsaw and Kharkov