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*MATHEMATICS*
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# On Banach—Mazur Distance between Certain Minkowski Spaces

by

V. I. GURARII, M. I. KADEC and V. I. MACAEV

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Let $E_1$ and $E_2$ be isomorphic Banach spaces. Let us consider a quantity

$$d(E_1, E_2) = \operatorname*{int}_T \|T\| \cdot \|T^{-1}\|,$$

where $T$ runs over all isomorphisms $E_1$ onto $E_2$. Banach and Mazur introduced a quantity $\ln d(E_1, E_2)$ (see [1]) which is the metric in any set of Banach spaces isomorphic in pairs with identified almost isometric spaces.*)

In this note we announce an estimation for $d(E_1, E_2)$, where $E_1, E_2$ are some Minkowski spaces (i.e. finite dimension Banach spaces), in particular the spaces $l_p^n$.**) We apply the obtained results to estimation of the projection constants ***) of Minkowski spaces. For results in this direction see [2], [3].

THEOREM 1. *If* $1 \leqslant p_1 \leqslant \infty$, $1 \leqslant p_2 \leqslant \infty$, $\operatorname{sign}(2-p_1) = \operatorname{sign}(2-p_2)$ *then*

$$d(l_{p_1}^n, l_{p_2}^n) = n^{\left|\frac{1}{n_1} - \frac{1}{p_2}\right|}.$$

THEOREM 2. *If* $1 \leqslant p_1 \leqslant 2 \leqslant p_2 \leqslant \infty$ *then for any* $n = 1, 2, \ldots$ *the following inequality takes place*

$$\max\left\{2^{\frac{2-p_2}{p_2}} n^{\frac{1}{p_1}-\frac{1}{2}}, 2^{\frac{p_1-2}{p_1}} n^{\frac{1}{2}-\frac{1}{p_2}}\right\} \leqslant d(l_{p_1}^n, l_{p_2}^n) \leqslant \max\left\{C_{p_2,n} n^{\frac{1}{p_1}-\frac{1}{2}}, C_{p_1,n} n^{\frac{1}{2}-\frac{1}{p_2}}\right\},$$

*where*

$$C_{p,n} = \begin{cases} 1 & \text{for} \quad n = 2^k \quad k = 0, 1, 2, \ldots, \\ (1+\sqrt{2})^{\frac{|2-p|}{p}} & \text{for all remaining} \quad n. \end{cases}$$

*) Banach spaces $E_1$ and $E_2$ are said to be almost isometric, if $d(E_1, E_2) = 0$ [1].

**) $l_p^n$ $(1 \leqslant p < \infty)$ (resp. $l_\infty^n = C^n$) is the space of sets consisting of $n$ real numbers $\{\xi_i\}_{i=1}^n$ with vector operations defined in a natural way and with the norm $\|\{\xi_i\}_{i=1}^n\| = (\sum_{i=1}^n |\xi_i|^p)^{\frac{1}{p}}$ (resp. $\|\{\xi_i\}_{i=1}^n\| = \max_{1 \leqslant i \leqslant n} |\xi_i|$).

***) Projection constant of Banach space $P$ is a quantity $\lambda(P) = \sup_B \lambda(P, B)$, where $B$ runs over all Banach spaces containing $P$ as subspace, and $\lambda(P, B) = \operatorname*{int}_A \|A\|$, where $A$ runs over all projects from $B$ onto $P$.

For any Banach spaces $E_1$ and $E_2$ it is easy to establish the inequality ([2]):

$$\lambda(E_1) \leqslant \lambda(E_2)\, d(E_1, E_2). \tag{1}$$

From (1) and from Theorems 1—2 follows

THEOREM 3. *If* $2 \leqslant p \leqslant \infty$ *then*

$$\lambda(l_p^n) \leqslant n^{\frac{1}{p}}.$$

*If* $1 \leqslant p < 2$ *then*

$$\lambda(l_p^n) \leqslant C_{p,n} \sqrt{n},$$

*where* $C_{p,n}$ *has the same value as in Theorem* 2.

Remark. B. Grünbaum [3] found the exact value of $\lambda(l_1^n)$:

$$\lambda(l_1^n) = 2^{1-n}\, n C_{n-1, \frac{n-2}{2}}. \tag{2}$$

DEFINITION 1. Let $\{e_i\}_{i=1}^n$ be a basis *) in Minkowski space $B$, $\dim B = n$. The quantity

$$\varkappa(\{e_i\}_{i=1}^n) = \sup_{\{a_i\}_{i=1}^n,\ \varepsilon_i = \pm 1,} \frac{\left\|\sum_{i=1}^n \varepsilon_i a_i e_i\right\|}{\left\|\sum_{i=1}^n a_i e_i\right\|}$$

is said to be the *coordinate asymmetry* of $\{e_i\}_{i=1}^n$. The quantity

$$\delta(\{e_i\}_{i=1}^n) = \sup_{\{a_i\}_{i=1}^n,\ \{e_i'\}_{i=1}^n} \frac{\left\|\sum_{i=1}^n a_i e_i'\right\|}{\left\|\sum_{i=1}^n a_i e_i\right\|},$$

where $\{e_i'\}_{i=1}^n$ denotes a permutation of $\{e_i\}_{i=1}^n$, is said to be the *diagonal asymmetry of* $\{e_i\}_{i=1}^n$. The quantity $\alpha(\{e_i\}_{i=1}^n) = \varkappa(\{e_i\}_{i=1}^n)\, \delta(\{e_i\}_{i=1}^n)$ is said to be the *asymmetry* of $\{e_i\}_{i=1}^n$.

DEFINITION 2. Let $\mathfrak{L}$ be the set of all bases in Minkowski space $B$. We shall call *coordinate asymmetry* (resp. *diagonal asymmetry*, *asymmetry*) of $B$ the quantity

$$\varkappa(B) = \inf_{\{e_i\}_{i=1}^n \in \mathfrak{B}} \varkappa(\{e_i\}_{i=1}^n),$$

respectively

$$\delta(B) = \inf_{\{e_i\}_{i=1}^n \in \mathfrak{B}} \delta(\{e_i\}_{i=1}^n), \qquad \alpha(B) = \inf_{\{e_i\}_{i=1}^n \in \mathfrak{B}} \alpha(\{e_i\}_{i=1}^n).$$

We shall call the space $B$ *coordinate symmetrical* (resp. *diagonal symmetrical*, *symmetrical*) if $\varkappa(B) = 1$, (resp. $\delta(B) = 1$, $\alpha(B) = 1$).

It is easy to show that in the given definitions we may instead of "sup" and "inf" write "max" and "min", respectively.

*) We regard all bases considered here as normalized, i.e. $\|e_i\| = 1$, $i = 1, 2, \ldots, n$.

THEOREM 4. *Let $B$ be an $n$-dimensional Minkowski space and let for certain basis $\{e_i\}_{i=1}^n$ in $B$:*

$$\varkappa(\{e_i\}_{i=1}^n) = \varkappa, \quad \delta(\{e_i\}_{i=1}^n) = \delta.$$

*Then the following inequality*

$$d(B, c^n)\, d(B, l_1^n) \leqslant \tfrac{1}{2}\varkappa(\varkappa+1)\,\delta n \tag{3}$$

*is true. The constant $\frac{1}{2}$ in (3) is exact.*

COROLLARY 1. *If $B$ is an $n$-dimensional symmetrical Minkowski space then the following inequality takes place:*

$$d(B, c^n)\, d(B, l_1^n) \leqslant n.$$

THEOREM 5. *Let for a certain basis $\{e_i\}_{i=1}^n$ in $n$-dimensional Minkowski space $B$: $\varkappa(\{e\}_{i=1}^n) = \varkappa$, $\delta(\{e_i\}_{i=1}^n) = \delta$.*
*Then*

$$d(B, c^n) \leqslant \sqrt{\frac{\varkappa(\varkappa+1)\,\delta n d(l_1^n, c^n)}{2}}.$$

COROLLARY 2. *If $B$ is an $n$-dimensional symmetrical Minkowski space, then*

$$d(B, c^n) \leqslant \sqrt{n d(l_1^n, c^n)}.$$

From this Corollary and from Theorem 2 follows

COROLLARY 3. *If $B$ is an $n$-dimensional symmetrical Minkowski space, then $\lambda(B) \leqslant d(B, c^n) \leqslant D_n n^{3/4}$, $n = 1, 2, \ldots$,*
*where*

$$D_n = \begin{cases} 1 & \text{for} \quad n = 2^k, \quad k = 0, 1, 2, \ldots \\ \sqrt{1+\sqrt{2}} & \text{for all remaining} \quad n. \end{cases}$$

Remark. For an arbitrary $n$-dimensional Minkowski space $B$ the inequality

$$d(B, c^n) \leqslant n$$

is well-known; it follows from John's inequality [4]

$$d(B, l_2^n) \leqslant \sqrt{n}.$$

THEOREM 6. *Let for certain basis $\{e_i\}_{i=1}^n$ in $n$-dimensional Minkowski space $B$:*

$$\varkappa(\{e_i\}_{i=1}^n) = \varkappa, \quad \delta(\{e_i\}_{i=1}^n) = \delta.$$

*Then*

$$\lambda(B) \leqslant \sqrt{\frac{\varkappa(\varkappa+1)\,\delta n \lambda(l_1^n)}{2}}.$$

From (2) and from Theorem 6 follows

COROLLARY 4. *If $B$ is an $n$-dimensional symmetrical Minkowski space, then*

$$\lambda(B) \leqslant 2^{\frac{1-n}{2}} n \sqrt{C_{n-1, \frac{n-2}{2}}}.$$

PROBLEMS. Does a sequence of Minkowski space $\{B_n\}_{n=1}^{\infty}$, $\dim B_n = n, n = 1, 2, \ldots$ exist, such that

$$\lim_{n\to\infty} \varkappa(B_n) = \infty?, \quad \lim_{n\to\infty} \delta(B_n) = \infty?, \quad \lim_{n\to\infty} \alpha(B_n) = \infty?.$$

MECHANICAL AND MATHEMATICAL FACULTY, UNIVERSITY OF KHARKOV (USSR)

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